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On Friedmann-Lemaître-Robertson-Walker cosmologies in non-standard gravity

Diego Sáez Gómez

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Dr. Emilio Elizalde Rius and Dr. Sergei D. Odintsov

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A mis padres, Dionisio y Teresa

Resumen

La presente memoria de tesis tiene como objeto el estudio de soluciones cosmológicas descritas por una métrica FLRW (en general tomada espacialmente plana) en el contexto, fundamentalmente, de extensiones de la Relatividad General, pero también en presencia de algunos de los candidatos más populares de la energía oscura.

Prácticamente desde los tiempos de Copérnico, la noción fundamental sobre nuestro Universo es la ausencia de una posición privilegiada de cualquier observador, lo que traducido en términos modernos da lugar al llamado *Principio Cosmológico*, que asume el hecho de que nos encontremos en un Universo homogéneo e isótropo, lo que traducido en palabras significa que la métrica es la misma en todo punto del espacio y que éste parece el mismo sea cual sea la dirección desde la que se observe. Esta aseveración, que puede parecer pretenciosa si tenemos en cuenta que el centro del Sol es muy diferente al medio intergaláctico, podría tener validez a escalas cosmológicas, es decir, mucho más allá de ni siquiera los cúmulos de galaxias. A pesar de que hoy en día hay sectores de la comunidad científica que cuestionan dicha validez, las observaciones de las galaxias lejanas y especialmente la gran isotropía del CMB, siguen mostrándose favorables a mantener el principio. El modelo de descripción del Universo, el llamado modelo del Big Bang, que tiene en dicho principio su base fundamental, ha sabido predecir con bastante acierto algunos de los aspectos más fundamentales de la evolución cosmológica, como puede ser la abundancia de los elementos. Dicho modelo predice un Universo en expansión que comenzó tras una gran explosión. Sin embargo, los detractores de dicho modelo cuestionan ciertos aspectos de la teoría, pues el modelo a su vez no da respuesta a problemas fundamentales como la *planitud* o el problema del horizonte (bien contrastados en las observaciones), para los cuales es necesario "parchear" la teoría con nuevos mecanismos (inflación en este caso), que a su vez introducen nuevos problemas. A pesar de todo, el modelo del Big Bang sigue siendo hoy la mejor descripción del Universo que poseemos, y los problemas que la teoría pueda contener, pueden ser vistos como la incompletitud de nuestras teorías, y podrían servir para descubrir nueva física que pudiera dar lugar a futuras predicciones.

En el año 1998, a través de observaciones de Supernovas IA, se llegó a la conclusión de que el Universo no solo estaba en expansión como había demostrado Hubble setenta años atrás, sino que además dicha expansión se estaba acelerando. Mucho antes de que éste descubrimiento tuviera lugar otro problema diferente sacudía el mundo de la física teórica, el llamado problema de la *constante cosmológica.* La cuestión aquí se debía a si la contribución de la densidad de energía de vacío que predecía la física cuántica debía tomarse en consideración en las ecuaciones de Einstein a la hora de estudiar un determinado sistema como el cosmológico. Dicha contribución aparecía en las ecuaciones en la forma precisamente de una constante, que a nivel cosmológico podía producir una expansión acelerada. Esto podría parecer el final feliz de un problema abierto, la aceleración del Universo no sería sino el resultado de la presencia de una constante cosmológica, cuyo origen estaría en el vacío cuántico. Sin embargo, lejos de resolver el problema, quizás lo agravó, pues la diferencia entre la densidad de vacío predicha por las teorías cuánticas, y aquélla medida en las observaciones es de 122 órdenes de magnitud!

Esto dió lugar a que en la última década se hayan propuesto multitud de posibles candidatos para explicar dicha aceleración del Universo, todos ellos englobados bajo el nombre de enegía oscura, siendo el más popular de estos modelos el llamado modelo Λ CDM, que sigue considerando la constante cosmológica como única responsable. Sin embargo, la constante cosmológica, además de los problemas ya expuestos, posee una ecuación de estado constante $p_\Lambda = -\rho_\Lambda$, mientras que las observaciones apuntan cada vez más a un carácter dinámico de dicha ecuación de la energía oscura. Esta época de expansión acelerada comenzaría cuando la densidad de materia bariónica/oscura descendiese, y la de la energía ocura pasaría entonces a dominar.

Sin embargo, además de esta época de aceleración, es necesario considerar la inflación también, la cual supone otra fase de aceleración al comienzo del Universo, y cuyo principal candidato es el campo escalar

conocido como inflatón. Todo ello hace pensar de forma natural que ambas fases pudieran estar relacionadas de algún modo, y que ambas hayan sido el producto de un mismo mecanismo.

En este contexto de cosmología actual, la presente tesis tiene como objetivo fundamental la reconstrucción de modelos cosmológicos que a través de un mismo mecanismo puedan explicar tanto la inflación como la aceleración actual del Universo, y dar una respuesta fehaciente la evolución del Universo. Para ello, a partir de soluciones FLRW y haciendo uso de las ecuaciones de campo, se reconstruirá la acción precisa que reproduce una solución determinada. En este sentido se investigará en primer lugar campos escalares del tipo "quintessence/phantom", los cuales ya han sido considerados como candidatos para energía oscura, pero que aquí también se considerará la posibilidad de que puedan servir para explicar inflación. Varios ejemplos de esta reconstrucción se muestran en el capítulo dos. Además, aquí también se considerará la posibilidad de utilizar varios escalares de este tipo para poder producir una transición suave a una fase de "phantom" (donde la ecuación de estado sería menor que -1), pues dicha transición se ha demostrado inestable cuando hay solo un campo escalar cuyo término cinético se hace cero en algún punto. Por otro lado, se explorarán teorías del tipo Brans-Dicke, donde la gravedad viene definida por el campo métrico y por un campo escalar fuertemente acoplado, estas teorías tienen un gran interés, pues aparte del hecho de que puedan explicar la energía oscura, son equivalentes a las llamadas teorías de gravedad modificada $f(R)$, lo cual supone una buena herramienta para estudiar éstas últimas tal y como se explorará en el capítulo cuatro. Además, se estudiará la relación, siempre discutida, entre estas teorías escalar-tensor definidas en el marco de Jordan y el de Einstein, donde interesantes resultados serán obtenidos. Se explora también la posibilidad de un Universo con una evolución oscilatoria, donde el llamado problema de la coincidencia (el hecho de que las densidades de materia y energía oscura sean prácticamente iguales en el presente, y no antes ni después) puede ser fácilmente rebatido, puesto que el Universo se movería por fases cíclicas.

Al margen de esto, el principal objetivo de la memoria es el estudio de teorías de gravedad modificada, especialmente las llamadas teorías $f(R)$ y gravedad de Gauss-Bonnet, como respuestas a la energía oscura conjuntamente con inflación. Dado que los principios que sustentan la Relatividad General, el de equivalencia, covariancia y relatividad, no hacen una gran restricción a la forma que han de tener las ecuaciones de campo, la elección de éstas puede quedar supeditada a las observaciones y experimentos. Ya en su momento Einstein eligió sus ecuaciones por el hecho de que, aparte de la conservación de la energía, en el límite no relativista se recuperaba la expresión de Newton. Del mismo modo, uno puede pensar que la energía oscura y la inflación no son más que malas definiciones de nuestras ecuaciones que, bajo ciertas correcciones, pueden ser explicadas. En este sentido se mostrarán diversas reconstrucciones de teorías que puedan explicar las métricas de FLRW que se propongan. Especialmente se hará énfasis en la reconstrucción del modelo de Λ CDM sin constante cosmológica pero con términos geométricos adicionales, así como de modelos donde exista una transición "phantom". Además, este tipo de teorías podrían dar una explicación certera al problema de la constante cosmológica, pues la energía de vacío es contrarrestada por los términos geométricos, dando lugar al valor observado. También se estudiarán modelos donde fluidos adicionales son considerados conjuntamente con las correcciones de las ecuaciones para explicar los comportamientos de la evolución del Universo. En este contexto se explorará también la relación entre los marcos de Jordan y el de Einstein.
Sin embargo, hay que ser cuidadoso a la hora de modificar dichas ecuaciones, pues tales correcciones pueden dar resultados exitosos a escalas cosmológicas pero catastróficas violaciones a escalas locales como la Tierra o el Sistema Solar. En este sentido se estudiarán los llamados modelos "viables o realistas", donde las modificaciones de la Relatividad General solo son importantes a grandes escalas, y la ley de Einstein se recupera a escalas menores, un poco en la misma línea que el mecanismo de "camaleón" en las teorías escalar-tensor. Por tanto, se estudiará la reconstrucción de soluciones cosmológicas (espaciostiempo FLRW planos esencialemnte) en el marco de teorías $f(R)$ y Gauss-Bonnet, donde se mostrará como, de

forma natural, las épocas de inflación y aceleración actual, pueden ser explicadas en el marco de dichas modificaciones sin necesidad de invocar componentes exóticos.
Además, y dada la reciente propuesta por parte de Hořava de una teoría gravitatoria que parece ser renormalizable, distintos aspectos de ésta serán estudiados. Dicha teoría, conocida ya como gravedad de Hořava-Lifshitz, asume una anisotropía en la coordenada temporal respecto a las espaciales, de manera similar a las anisotropías utilizadas en las teorías de Lifshitz en materia condensada, lo que supone la renormalización de la teoría pero a la vez la ruptura de la covariancia, dando lugar a violaciones de los principios que sustentan (y que están bien comprobados por los experimentos a escalas muy por debajo de la escala de Planck) la relatividad de Einstein. Sin embargo, se supone que en el límite infrarojo (IR) la teoría clásica se recupera, siendo el mecanismo para ello no del todo claro, aunque se han realizado intentos para comprenderlo. El último de ellos es introducir una nueva simetría $U(1)$ en el modelo, lo que forzaría en el límite IR a recuperar la Relatividad General, en este sentido en la presente memoria se exploran extensiones de dicho modelo. Otro modo, estudiado en la presente tesis, y que evita las inestabilidades del modo de espín-0 que la teoría de Hořava produce en el límite IR alrededor de la solución de vacío de la teoría, es considerar una acción, donde "de Sitter" sería estable y considerada como la solución natural en vacío en lugar del espaciotiempo de Mikowski.
Al margen de los problemas intrínsecos de la teoría, la gravedad de Hořava-Lifshitz sufre del mismo problema que la Relatividad General a nivel cosmológico, esto es que no puede explicar por sí sola la aceleración del Universo, tanto actual como de la época inflacionaria. De modo que en analogía con las extensiones estudiadas en gravedad $f(R)$, la acción de Hořava-Lifshitz puede extenderse a acciones más generales $f(\tilde{R})$ para reconstruir soluciones cosmológicas que reproduzcan los efectos de la energía ocura, y que incluso lo unifiquen con inflación. En este contexto, varios modelos y ejemplos son estudiados. Además se explorará las correcciones Newtonianas debidas al modo escalar introducido por la función $f(\tilde{R})$ (que no por la anisotropía temporal), y la reconstrucción de una acción libre de singularidades cosmológicas.

Por último, respecto a las singularidades, los dos últimos capítulos de esta memoria estarán dedicados al estudio de posibles efectos semiclásicos alrededor de ellas. Las singularidades son bastante comunes en soluciones de la Relatividad General, así como en sus extensiones, y suponen puntos del espaciotiempo donde nuestras teorías se suponen no válidas y es necesario una teoría cuántica consistente de la gravedad. Algunos efectos de origen cuántico como el efecto Casimir o la anomalía conforme, serán explorados alrededor de singularidades cosmológicas del tipo "Big Rip". Se demostrará que dichos efectos semiclásicos apenas tienen relevancia y que la singularidad no es evitada en la mayor parte de los casos. Además, se estudiará la validez de los límites sobre la entropía del Universo, como aquéllos propuestos por Verlinde, y la validez de la fórmula de Cardy-Verlinde, que da una derivación directa de los límites dinámicos de la entropía, y la correspondencia que se establece entre las ecuaciones de FLRW y las de una Teoría de Campos Conforme (CFT) 2d. Todo ello será estudiado en contextos más generales.

Resum

La present memòria de tesis té com a objecte l'estudi de solucions cosmològiques descrites per una mètrica FLRW (en general espacialment plana) en el context, fonamentalment, d'extensions de la Relativitat General, però també en presència d'alguns dels candidats més populars de l'energia fosca.
Practicament des dels temps de Copèrnic, la noció fonamental sobre al nostre Univers és l'absència d'una posició privilegiada de qualsevol observador, el que en termes moderns dóna lloc a l'anomenat *Principi Cosmològic*, que assumeix el fet que ens trobem en un Univers homogeni i isòtrop, el que traduït en paraules planeres vol dir que la mètrica és la mateixa en tot punt de l'espai i que aquest sembla el mateix sigui la que sigui la direcció des de la qual s'observi. Aquesta asseveració, que pot semblar pretenciosa si tenim en compte que el centre del Sol és molt diferent al medi intergalàctic, podria tenir validesa a escales cosmològiques, és a dir, molt més enllà dels cúmuls de galàxies. Tot i que avui en dia hi ha sectors de la comunitat científica que qüestionen la seva validesa, les observacions de les galàxies llunyanes i especialment la gran isotropia del CMB, segueixen mostrant-se favorables a mantenir el principi. El model de descripció de l'Univers, l'anomenat model del Big Bang, que té en aquest principi la seva base fonamental, ha sabut predir amb força encert alguns dels aspectes més fonamentals de l'evolució cosmològica, com pot ser l'abundància dels elements. Aquest model prediu un Univers en expansió que va començar després d'una gran explosió. No obstant això, els detractors d'aquest model qüestionen certs aspectes de la teoria, ja que el model no dóna resposta a problemes fonamentals com la *planitud* o el problema de l'horitzó (ben contrastats a les observacions), per als quals cal "apedaçar" la teoria amb nous mecanismes (inflació en aquest cas), a la vegada introdueixen nous problemes. Malgrat tot, el model del Big Bang segueix sent avui la millor descripció que posseïm de l'Univers, i els problemes que la teoria pugui contenir, poden ser entesos com deguts a la imcompletesa de les nostres teories, i podrien servir per descobrir nova física que pogués donar lloc a futures prediccions.
L'any 1998, a través d'observacions de Supernoves IA, es va arribar a la conclusió que l'Univers no només estava en expansió, com havia demostrat Hubble setanta anys abans, sinó que a més aquesta expansió s'estava accelerant. Molt abans que aquest descobriment tingués lloc, un altre problema diferent sacsejà el món de la física teòrica, l'anomenat problema de la constant cosmològica. La qüestió era si la contribució de la densitat d'energia del buit que predeia la física quàntica s'havia de tenir en compte en les equacions d'Einstein a l'hora d'estudiar un determinat sistema com el cosmològic. Aquesta contribució apareixia en les equacions en la forma precisament d'una constant, que a nivell cosmològic podia produir una expansió accelerada. Això podria semblar el final feliç d'un problema obert, l'acceleració de l'Univers no seria sinó el resultat de la presència d'una constant cosmològica, amb origen en el buit quàntic. No obstant això, lluny de resoldre el problema, més aviat el va agreujar, ja que la diferència entre la densitat de buit predita per les teories quàntiques, i la mesurada per les observacions, és de 122 ordres de magnitud!
Això va donar lloc a que en l'última dècada s'hagin proposat multitud de possibles candidats per explicar aquesta acceleració de l'Univers, tots ells englobats sota el nom d'energia fosca, sent el més popular d'aquests models l'anomenat model Λ CDM, que segueix considerant la constant cosmològica com a única responsable de l'acceleració. No obstant això, la constant cosmològica, a més dels problemes ja exposats, té una equació d'estat constant $p_\Lambda = -\rho_\Lambda$, mentre que les observacions apunten cada vegada més a un caràcter dinàmic d'aquesta equació de l'energia fosca. L'època d'expansió accelerada començaría quan la densitat de matèria bariònica/fosca baixés, i la de l'energia fosca passaria llavors a dominar.
Però, a més d'aquesta època d'acceleració, cal considerar la inflació també, que suposa una altra fase d'acceleració al començament de l'Univers, on el principal candidat és el camp escalar conegut com inflató. Tot això fa pensar de forma natural que les dues fases puguin estar relacionades d'alguna manera, i que ambdues hagin estat el producte d'un mateix mecanisme.

En el context de la cosmologia actual, aquesta tesi té com a objectiu fonamental la reconstrucció de models cosmològics que mitjançant un únic mecanisme puguin explicar tant la inflació com l'acceleració actual de l'Univers, i donar una resposta fefaent a l'evolució de l'Univers. Per això, a partir de solucions FLRW i fent ús de les equacions de camp, es reconstruirà l'acció precisa que reprodueix una solució determinada. En aquest sentit s'investigarà en primer lloc camps escalars del tipus "quintessence/phantom", els quals ja han estat considerats com a candidats per a l'energia fosca, però que també es considerarà la possibilitat que puguin servir per explicar inflació. A més, aquí també es considerarà la possibilitat d'utilitzar diversos escalars d'aquest tipus per poder produir una transició suau a una fase de "phantom" (on l'equació d'estat seria menor que -1), ja que aquesta transició s'ha demostrat inestable quan hi ha només un camp escalar, el terme cinètic es fa zero en algun punt. D'altra banda, s'exploren teories del tipus Brans-Dicke, on la gravetat ve definida pel camp mètric i per un camp escalar fortament acoblat, aquestes teories tenen un gran interès, perquè a banda del fet que puguin explicar l'energia fosca, són equivalents a les anomenades teories de gravetat modificada $f(R)$, la qual cosa suposa una bona eina per estudiar aquestes últimes tal com s'explorarà en el capítol quatre. A més, s'estudiarà la relació, sempre discutida, entre aquestes teories escalar-tensor definides en el marc de Jordan i el d'Einstein, on obtindrem interessants resultats. S'explorarà també la possibilitat d'un Univers amb una evolució oscilatoria, on l'anomenat problema de la coincidència (el fet que les densitats de matèria i energia fosca siguin pràcticament iguals en el present, i no abans ni després) pot ser fàcilment rebatut, ja que l'Univers es mouria per fases cícliques.

Al marge d'això el principal objectiu de la memòria és l'estudi de les teories de gravetat modificada, especialment les anomenadas teories $f(R)$ i la gravetat de Gauss-Bonnet, com a resposta a la energía fosca conjuntament amb inflació. Donat que els principis que sustenten la relativitat general, els d'equivalència, covariància i relativitat, no imposen una gran restricciò sobre la forma que han de tenir les equacions de camp, l'elecció d'aquestes pot quedar supeditada a les observacions i experiments. Cal recordar que Einstein va escollir les ecuacions pel fet que, a part de la conservació de la energia, en el limit no relativista hom recuperava la expressió de Newton. De la mateixa manera, un pot pensar que l'energía fosca i la inflació no son més que males definicions de la teoria que, sota certes correccions, poden ser explicades. En aquest sentit, es mostraran diverses reconstruccions de teories que puguin explicar per si soles les metriques que es proposin. Especialment, es posara énfasis en la reconstrucció del model Λ CDM sense constant cosmològica pero amb termes geometrics adicionals, així com de models on existeixi una transició "phantom". A més, aquest tipus de teories podrien donar una explicació precisa al problema de la constant cosmològica, doncs en aquest cas l'energia de buit es contrarestada pels termes geomètrics donant lloc al valor observat. També s'estudiaran models on fluids adicionals son considerats conjuntament amb les correccions de les ecuacions, per explicar els comportaments de l'evolució de l'Univers. En aquest context, s'explorarà també la relació entre els marcs de Jordan i el d'Einstein. No obstant, cal ser curós a l'hora de modificar aquestes ecuacions, doncs les dites correccions poden donar resultats exitosos a escales cosmològiques pero catastròfiques violacions a escales locals com la Terra o el Sistema Solar. En aquest sentit, s'estudiaran els anomenats "models viables", on les modificacions de la Relativitat General només son importants a gran escala i la llei d'Einstein es recupera a escales menors, en la mateixa línia que l'anomenat "mecanisme de camaleó". S'estudiarà la reconstrució de solucions cosmològiques en el marc de teories $f(R)$, on es mostrarà com de forma natural les épogues d'inflació i d'acceleració actual poden ser explicades sense necessitat d'invocar continguts exòtics.
A més, i donada l'última proposta per part de Hořava d'una teoria gravitatòria que sembla ser renormalizable, n'estudiarem diferents aspectes. Aquesta teoria anomenada com gravetat de Hořava-Lifshitz, assumeix una anisotropía en la coordenada temporal respecte a les espacials, de manera similar a les anisotropies utilitzades en les teories de Lifshitz en matèria condensada, lo que suposa la renormalització de la teoria, però a l'hora la ruptura de la covariància donant lloc a violacions dels principis que sustenten la Relativitat. No obstant, es suposa que en el limit IR, la teoria clásica emergeix. El mecanisme no és,

per aixó, del tot clar, tot i que s'han realitzat diferents intents d'explicar-lo. L'ùltim d'aquests és introduir una nova simetria en el model, que forçaria en el limit IR a recuperar la Relativitat General. En aquest sentit, extensions d'aquest model son explorades en la present memòria. Una altra posible solució que podría evitar les inestabilitats del mode d'espí-0 que es produeix en el limit IR al voltant de la solució del buit de la teoria, seria considerar una acció, on "de Sitter" fos estable i que seria considerada com la solució natural al buit en lloc de Mikowski.
Al marge dels problemes intrínsecs de la teoria, la gravetat de Hořava pateix del mateix problema que la Relativitat General a nivell cosmològic, això és que no pot explicar l'acceleració del Univers, tant l'actual com la de l'epoca inlaccionària. Així doncs, en analogía amb les extensions estudiades de la Relativitat General, l'acció es pot estendre a $f(\tilde{R})$ per reconstruir solucions cosmològiques que reprodueixin els efectes de l'energía fosca i l'unifiquin amb inflació. Diferents models i exemples son estudiats, a més s'exploraran les correccions newtonianes (només per al mode escalar de $f(\tilde{R})$, i no per a la correcció de l'anisotropia temporal) i la reconstrucció d'una acció lliure de singularitats cosmològiques.
Per ùltim, respecte a les singularitats, els dos darrers capítols d'aquesta memòria estan dedicats a l'estudi de possibles efectes semiclàsics al voltant d'elles. Les singularitats són bastant comuns en solucions de la Relativitat General, així com en les seves extensions, i suposen punts de l'espaitemps, on les nostres teories es suposen no vàlides. Alguns efectes d'origen quàntic com l'efecte Casimir, o l'anomolia conforme seran explorats al voltant de singularitats cosmològiques del tipus "Big Rip". Es demostrarà que els efectes esmentats, gairebé no tenen rellevància i la singularitat no és evitada en la major part dels casos. A més, s'estudiarà la validesa dels límits sobre l'entropia de l'Univers al voltant d'aquestes singularitats. També s'estudiarà la fórmula de Cardy-Verlinde, que dóna una derivació directa dels limits dinàmics de l'entropia, i la validesa de la correspondència que s'estableix entre les equacions de FLRW i les d'una Teoria de Camps Conforme (CFT) en 2d, en contextes més generals.

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Some people say, "How can you live without knowing?" I do not know what they mean. I always live without knowing. That is easy. How you get to know is what I want to know.

Richard Feynman

La paciència comença amb llàgrimes i, al final, somriu.

Ramon Llull

The present thesis, aimed to obtain the title of Philosophy Doctor in Physics, is based on the following papers published in referred journals, pre-prints and conference proceedings: [51, 58, 126, 132, 135, 136, 138, 161, 191, 213, 242, 264, 265, 266, 267, 268, 269].

Contents

Chapter 1

Introduction

1.1 Modern cosmology: from Hubble law to dark energy

Let us start with a brief historical review of the main facts and discoveries occurred during the last century in the field of cosmology and gravitational physics. The theories constructed and observations performed along the last one hundred years, are the basis of the research nowadays. The history of Cosmology as a science can be looked as the construction of a pyramid that you do not know how high will be at the end, and whose basis has to be so robust to support even if increased. Thus, step by step cosmology is being built, where new problems arise on each step, so that the theory has to be amplified (sometimes just papered over it) with new elements in order to achieve the explanations of the problems arisen, what usually drive us to new physical aspects. The main question, probably with no right answer at present, is how strong is the basis in order to resist the weight to add new aspects and therefore, new problems. Among other questions, some of the main task of theoretical physics nowadays has something to be with cosmology, whose unresolved problems have motivated, among a lot of published papers, a lot of PhD thesis as the current one. This justifies the need to understand well the origins of modern cosmology, whose theory, observations and experiments are the basis of the research nowadays[1].

One could say that modern cosmology started with the appearance of Einstein theory of gravity in 1915, Ref. [127, 128], which abandoned the notion of gravity as a force, and instead of it, the so-called General Relativity (GR) proposed a geometrical point of view for gravity. As at present, one believed in the validity of the so-called *Cosmological Principle*[2], which basically establishes that the Universe is the same in every point (at large scales), what from a mathematical point of view, it means that the Universe is homogeneous and isotropic. Nevertheless, the first attempt to construct a realistic cosmological model in the frame of GR was based on a very different idea from what we think now, it was believed on an static Universe with no beginning and end. In order to achieve such kind of solution, Einstein introduced for the first time, the cosmological constant (cc) in his field equations (see Ref. [129]), what yields, under certain conditions, an unstable static Universe (what for Einstein would be *my biggest blunder of my life*). At the same time, Willem de Sitter found an expanding cosmology with the presence of a cosmological constant, but this kind of cosmology was ignored along the years. Later in 1922, Alexander Friedmann found a solu-

[1] On modern cosmology, see for example the textbooks Ref. [210, 253, 254].

[2] Actually at that moment it was believed on the validity of Copernican Principle, and it was not until 1933 when modern Cosmological Principle was proposed by Edward Milne, which supposed a generalization of the preceding one.

tion of the Einstein's field equations that suggests an expanding Universe. Almost simultaneously, Georges Lemaitre proposed for the first time, a creation event as the beginning of the Universe expansion, being the first model that later would be known as the Big Bang model[3], and proposed the distance-redshift relation that would explain the expansion of the space. These proposals, together with the metric given by Howard Percy Robertson and Arthur Geoffrey Walker, give the name of what we know nowadays as the Friedmann-Lemaître-Robertson-Walker solution of the gravity equations, and which (we believe) is the best fit to describe our Universe evolution. However, the crucial fact occurred in 1929, when Edwin Hubble showed the linear relation between the distance and redshift[4]. By using the observational data accounted by Vesto Slipher on galaxies spectra some years before, he showed the fact of the expansion of the Universe.

From here on, the discover of the expansion of the Universe gave a strong support to the development of the Big Bang model proposed by Lemaitre. The measures on the abundances of the elements as the Hydrogen, Helium or Lithium were well predicted by the Big Bang model, by means of the initial hot state of the Universe, when primordial nucleosynthesis occurred. As the Universe was growing, it became colder and colder, such that the radiation decoupled, and at that moment, it was emitted what we call the Cosmic Microwave Background (CMB), discovered by the astronomers Arno Penzias and Robert Wilson in 1965. Then, these two facts among others, made the Big Bang model as the most popular model to explain the Universe evolution[5]. However, this does not mean that the model was absent of problems. Actually, during the sixties and seventies, main unresolved questions of the model were pointed out:

- The horizon problem, which present the problem of the causal connection between far away sides of the Universe.
- The flatness problem. Big Bang model does not predict a flat Universe while the observations on the CMB suggested that the Universe was almost flat.
- Other important problems: Baryogenesis, the observed asymmetry between matter-antimatter is absent of any explanation. The Monopole problem, it consists on the production of magnetic monopoles at the initial states of the Universe, which is not observed.

These were probably the main troubles of the Big Bang model at that time. However, the horizon and flatness problems could be explained at the beginning of the eighties by postulating an initial super-accelerated phase called inflation, proposed initially by Alan Guth in 1981, Ref. [169] (what nowadays is known as old inflation), and later by Andrei Linde in 1982, Ref. [201]. While the baryogenesis problem is still unresolved (for a review on the topic see Ref. [125]), although I will not enter on this trouble as this is not the target of the present thesis. Almost simultaneously, it was pointed out that the rotational curves of galaxies could not be explained in the frame of GR if no new and unobserved matter is included in our galaxies and then, in our Universe. This new matter, which is believed to be ten times the amount of observed barionic matter, was called Dark Matter, and among the astrophysical problems, it is also necessary in order to explain the structure formation of the Universe, the CMB anisotropies or the observed gravitational lensing. Then, we have another question to resolve, to find out what actually Dark Matter is.

Then, we had with these ingredients, an inflationary-Big Bang model, with relatively successful, but not absent of serious problems. However, our pyramid hided another surprise inside, in 1998 by means of

[3]The term Big Bang was later proposed by Fred Hoyle in 1950.

[4]Hubble's law established a linear redshift-distance linear relation, which is valid to distance of some hundreds of Mpc, but has to be corrected at larger scales

[5]Note that an alternative to the Big Bang model, the so-called Steady State Theory, was important until the discovery of CMB in 1965 and definitely refused when the anisotropies of CMB were measured by COBE in 1992.

measuremens of the luminosity of Supernovae IA, it was discovered that the Universe is not just expanding but accelerating its expansion, what supposed the beginning of the so-called dark energy models in order to explain such atypical behavior of the Universe at large scales.

It is difficult to explain in a brief introduction all the implications of these (not essential) modifications of the standard Big Bang model. However, one has to note that inflation, late-time acceleration or CMB observations are products of the basis of our model, what means that under a different model probably we would have never reached, along the last century, such conclusions. Then, now the question arises, is our pyramid on the Big Bang model so robust to support all these troubles? while we try to answer this difficult question, let us try to study and find a natural explanation to the problems that the model still has.

Inflationary cosmology

As it has been commented above, inflation was needed in order to explain two intrinsic problems of the Standard model of cosmology: the horizon and flatness problems. It consists in an initial accelerated phase after the origin of the Universe, that makes the Universe expansion to proceed beyond its proper horizon. This epoch ends in a hot state, assumed to be produced by the so-called *reheating*, and leads into the known radiation and matter dominated epochs. The first attempt, pointed out by Alan Guth Ref. [169], was based on a first order transition phase of our Universe from a false vacuum to a true one. The responsible for such transition is the so-called inflaton, which is immersed initially in a false vacuum, acting as an effective cosmological constant and producing a de Sitter accelerating expansion. Then, inflation ends when the inflaton tunnels to the true vacuum state nucleating bubbles. The problem (already pointed by Guth in his original paper) is that in order to get an initial hot state after inflation ends, the nucleated bubbles should be enough close to produce radiation by means of collisions between the walls of the bubbles. However, as inflation has to last long enough to resolve the initial problems, the bubbles stay far from each other, and the collisions between bubbles become very rare, what gives no possibility to reheat the Universe and get the initial Big Bang state, where primordial nucleosyntesis has to be realized.

Nevertheless, after this initial model, it was Linde who proposed a more successful one, where inflation is driven by a scalar field ϕ, which produces a second order phase transition. In this model, known as slow-rolling model, inflation occurs because the potential of the scalar field $V(\phi)$ has a flat section, working as an effective cc, where the field slow rolls. Then, the potential is assumed to have an steep well where the inflaton decays starting to roll faster to a minimum situated in $V(\phi) = 0$, breaking the slow-roll and ending the inflationary period (an example of such kind of potentials is given in Fig.1.1). When the field reaches the minimum, it starts to oscillate around it, producing particles and reheating the expanded Universe. This can resolve the problems of the old inflation, and also gives an explanation of the origin of density perturbations by means of quantum fluctuations of the scalar field, and which later produce the formation of large scale structures as well as the anisotropies of the CMB. However, some questions are still unresolved, as the mechanism to start inflation (initial conditions problem) or how reheating works, what is not yet completely clear.

As it will be shown along the thesis, inflation can be also reproduced purely in terms of gravity, with no need to invoke scalar or other kind of fields. On this way, Starobinsky proposed in 1980 Ref. [285] a model for cosmology free of initial singularity, where higher order terms as R^2 are involved in the action. This model can reproduce inflation, what can be seen easily by the fact of the mathematical equivalence between some kind of scalar-tensor theories and modified gravities of the type $f(R)$. Nevertheless, the same problems as in scalar field inflation remain in modified gravity.

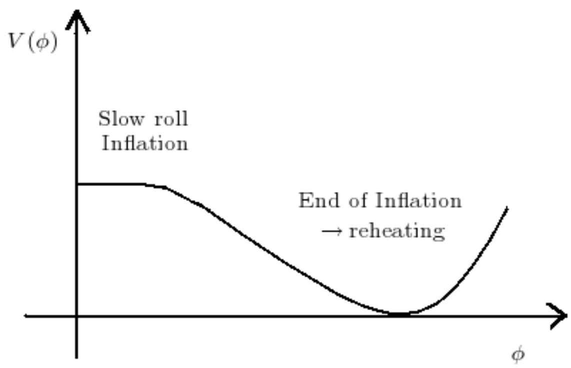

Figure 1.1: A typical potential for slow roll inflation. Other kind of potentials can also produce inflation

Another important aspect that will be studied in the present thesis is the possibility to unify inflation and late-time acceleration under the same mechanism (scalar fields, modified gravity or perhaps a combination of both). The majority of the scientific community, as a consequence of recent observations, has accepted that our Universe expansion is accelerated nowadays, so one could ask if both accelerated phases could be related, and produced by the same mechanism. In this way, in Chapter 2 and 3, the possibility to construct an scalar field potential that could reproduce both epochs is shown, as in Fig. 1.2, where the scalar field has decayed after inflation but the minimum is now non-zero such that the residual of inflation acts as a cc that becomes dominant at late-times when the radiation/matter densities decrease.

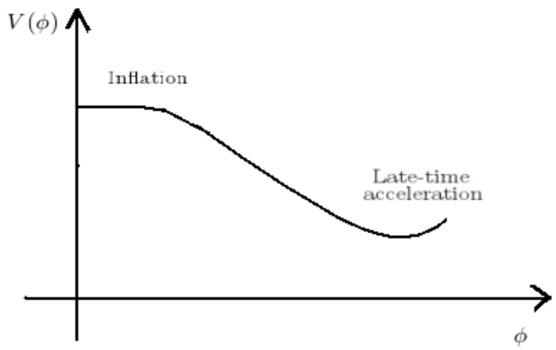

Figure 1.2: Example of an scalar potential for the unification of inflation and late-time acceleration.

In the same way, modified gravities as $f(R)$ or Gauss-Bonnet models, can reproduce both epochs in a natural way with no need of any extra field, and remaining invariant the rest of cosmological aspects, mainly the radiation/matter epochs, as it will be shown along the thesis.

Late-time acceleration

Since 1998, when observations of Supernova IA, realized by two independent groups (see Refs. [260, 257]), pointed to the idea of an accelerating expansion, such hypothesis has been accepted by the majority of scientific community. Since then, a lot of models and candidates, which are included under the name of

Dark Energy (DE), has been proposed in order to explain the phenomena (for reviews on DE candidates see Refs. [111, 227, 245, 249, 270]), being the Λ CDM model, where a cc acts as a perfect fluid with negative pressure, the most popular one.

Supernova data pointed to a transition from decelerated to accelerated expansion (assuming an spatially flat metric), such that a Universe with a current content of dust-matter $\Omega_m^0 = 1$ does not fit the observations, while it works when $\Omega_m^0 \sim 0.3$ and the rest of content is filled by a perfect fluid with negative pressure, actually with an Equation of State (EoS) $p = w\rho$, where $w < -1/3$ is required. This requires the violation of at least one of the energy conditions, which for a perfect fluid can be written as,

- Strong energy condition: $\rho + 3p \geq 0 \longrightarrow w \geq -1/3$.
- Weak energy condition: $\rho + p \geq 0 \longrightarrow w \geq -1$.

Then, in order to satisfy the requirement to have accelerating expansion, one has to violate at least the Strong energy condition, but may be also the weak one (Phantom case). As it has been mentioned, the simplest model that satisfied such requirement is a cc Λ, which behaves with an EoS $p_\Lambda = -\rho_\Lambda$. However, other candidates as scalar-tensor theories or modified gravities, analyzed along the present thesis, could be good candidates from a theoretical point of view, as it is shown below, or at least they could just contribute to the acceleration in combination with other candidates.
Before analyze the reasons to explore different models beyond Λ CDM, let us continue with the arguments

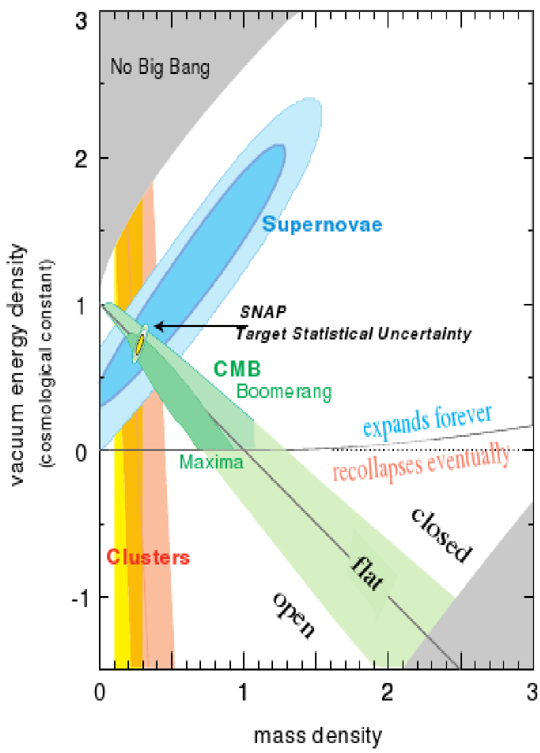

Figure 1.3: Confidence regions for dust-matter and dark energy densities constrained from the observations of Supernovae IA, CMB and galaxy clustering. From Ref. [3].

which support the idea of an accelerating Universe apart from the luminosity distance from Supernova.

Then, we have that the age of the Universe is very sensible to the content of the Universe, where the oldest stellar populations observed are about $13Gyrs$ old, so that the age of the Universe has to be $t_u > 13Gyrs$. Then, a Universe absent of dark energy violates such constraint, while for example for Λ CDM model, we have $t_u = 13.73Gyrs$. Also the observations related to the CMB and large scale structure support the idea of a Universe filled with dark energy. From the data collected by WMAP3, we have that the primordial fluctuations are nearly scale invariant, which agree very well with inflationary predictions, and the position of the first peak constraints the spatial curvature of the Universe to be $\Omega_K^0 = 0.015^{+0.02}_{-0.016}$ (see Ref. [283]), almost spatially flat, where it is assumed an EoS parameter for dark energy $w_{DE} = -1$. Then, the density for dark energy would be $\Omega_\Lambda = 0.72^{\pm}0.04$. In 1.3, the confidence areas are plotted using Supernova IA, CMB and large scale galaxy clustering. We can see that a non-flat Universe is almost refused and the most probable solution is a Universe filled with 30% of dust matter (barionic and cold dark matter) and 70% of the so-called Dark Energy.

Finally, it is important to note that all these measurements are sensible to the choice of model, such that in a FLRW Universe, the theory needs dark energy while in other kind of metrics, it may not.

Why not just Λ CDM?

As it was mentioned above, Λ CDM model was the first model proposed to explain dark energy and probably it is going to be the simplest (at least aesthetically) model shown at the present thesis. It is described by the action,

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}(R-2\Lambda) + \int d^4x\sqrt{-g}L_m \ , \tag{1.1}$$

where $g = \det[g_{\mu\nu}]$, being $g_{\mu\nu}$ the metric, R the Ricci scalar, L_m the lagrangian for the matter content (radiation, baryons, cold dark matter..), and Λ the cc. This action can well reproduced cosmic acceleration and fit the observational data commented above by fixing the value of Λ to be approximately the current Hubble parameter,

$$\Lambda \sim H_0^2 = (2.13h10^{-42})^2GeV^2 \longrightarrow \rho_\Lambda = \frac{\Lambda}{8\pi G} \sim 10^{-47}GeV^4 \ . \tag{1.2}$$

However, this presents several problems. The main one is the so-called *cosmological constant problem*[6], which was known much before late-time acceleration was discovered. In particle physics, the cc appears as a natural vacuum energy density, which has to be included in Einstein's field equation to contribute to gravity. The problem arises because of the large difference between the observed value of the cc (1.2), and the vacuum energy density predicted by particle physics, suffering of a severe fine-tuning problem. The vacuum energy density can be calculated as the sum of zero-point energies of quantum fields with mass m,

$$\rho_{vac} = S = \frac{1}{2}\int_0^\infty d^3k\sqrt{k^2+m^2} = \frac{1}{4\pi^2}\int_0^\infty dkk^2\sqrt{k^2+m^2} \propto k^4 \ , \tag{1.3}$$

which exhibits an ultraviolet divergence. Nevertheless, one expects that quantum field theory is valid up to a cut-off k_{max}, what makes the integral (1.3) converge. It is natural to take such cut-off for GR as the Planck scale m_{Pl}, what yields an vacuum energy density,

$$\rho_{vac} \sim 10^{74}GeV^4 \ . \tag{1.4}$$

That means 122 orders of magnitude larger than the observed value (1.3). Even if one takes the cut-off of scales for QCD, it is still very large compared with (1.3). There have been attempts to resolve this problem

[6]For a review on the "old" cc problem, see Ref [304].

by many and different ways (supersymmetry, strings, anthropic principle...), but there is not yet a successful answer, probably because of the absence of a consistent quantum theory of gravity. However, a possibility explored along this thesis may be a classical adjustment of the value of vacuum energy density, produced by scalar fields or extra gravity terms in the action, what may provide a natural relaxing mechanism of the large value (1.4) (see Refs.[279, 26]). The second question is not a theoretical problem as the one above, but it could be important in the future when more observational data are obtained. It is the question on the dynamics of the EoS for dark energy. For the case of a cc, we have an static EoS $p_\Lambda = -\rho_\Lambda$, while the EoS for dark energy may be dynamic and even inhomogeneous $p_\Lambda = w(t)\rho_\Lambda + f(t)$, what can be well modeled in scalar-tensor theories or modified gravities.

Another question that remained "unresolved" in the Λ CDM model, is the so-called coincidence problem, which establishes the need to explain why precisely nowadays the matter and dark energy densities are more or less equal (of the same order). Note that the quotation marks on the word "unresolved" is just to emphasize the fact that this problem could not be a real physical problem, but just the product of a real coincidence! For example, along the Universe history there was a period when radiation and dust matter densities were completely equal (during the transition between both eras). Then, if we were living during that moment, would we think of this as of a coincidence problem? probably we would not, in spite of the fact that, at the present, we look to such period as a natural evolution of our Universe. However, if we insist on the validity of the open question, we have to point out that a cc does not give a natural explanation of such coincidence, where another kind of candidates could contribute with a more natural answer, as for example an oscillating Universe (this possibility is explored in Chapter 3).

Finally, it is natural to relate, as it was mentioned above, the two accelerated epochs of the Universe evolution, on a unification of cosmic and inflationary eras under the same mechanism, what needs, apart from a vacuum energy density, another contribution from an scalar field, or extra terms in the gravitational field equations. This possibility is well explored and analyzed along the present report, as a real possibility that could resolve the problems commented above.

Probably the question of the dark energy, what adds up to the cc problem, is one of the main unresolved task of nowadays physics. As it was commented above, this question has produced a lot of publications where a plenty of different models have been proposed. Along this work, it will be analyzed, among other models, scalar-tensor theories and mainly modified gravities, exploring the difference between them. However, other attempts to resolve the mysteries of dark energy, which will not be analyzed here, have been considered in the literature, as for example with vectors (see [28]) or Yang-Mills fields (see Ref. [131]).

1.2 Scalar-tensor theories

Scalar-tensor theories have become the most popular ones, after Λ CDM, to model late-time acceleration. One of the main reasons is the simplicity to construct models using an scalar field, as it is well known that scalar-tensor theories can well reproduce any cosmology (see Ref. [139]). Also the success to reproduce inflation with an scalar field (the majority of models for inflation proposed during the eighties and nineties, contain an scalar field), makes from scalar fields a good candidates for dark energy. Another important reason, mentioned above, is that nowadays the observations do not exclude the possibility to have dark energy with a dynamical EoS, and not static as it occurs in Λ CDM. In the era of precision cosmology, the observations constraint the value of w to be close to -1 , but tell us little about the time evolution of w . In scalar-tensor theories usually the EoS parameter is going to be time-dependent, and it can be easily fixed to be close to -1 at the current epoch. Apart from the cosmological considerations, scalar fields naturally arise in particle physics as well as in string theory, where they play important roles, as the Higgs boson, in spite of what a spin-0 particle has never been detected.

A wide range of models have been proposed using scalar fields. In this thesis, we will analyze models with an action of the type,

$$S = \int dx^4 \sqrt{-g} \left[\frac{f(\phi)}{2\kappa^2} R - \frac{1}{2}\omega(\phi)\partial_\mu\phi\partial^\mu\phi - V(\phi) + L_m \right] \,, \tag{1.5}$$

where $f(\phi)$ determines the coupling between the scalar and metric field, $\omega(\phi)$ is the kinetic term and $V(\phi)$ the scalar potential. Depending on the values of (f, ω, V), the nature of the scalar field will be very different. However, scalar fields are not just restricted to the action (1.5), but there are other important theories that include other kind of scalar fields. Some of them are k-essence, which generalizes quintessence field and present a non canonical scalar field, the tachyon (a superluminal particle), or the Chaplygin gas, which also has been considered as a candidate to dark energy. Nevertheless, along the first part of the thesis, where scalar-tensor theories are analyzed, we will focus on scalar fields of the type quintessence and phantom, as well as we explore non-minimally coupling theories of the type of Brans-Dicke theory.

Note that quintessence or phantom behaviors can well be reproduced not just by an scalar field but also by other possible fields or modifications of gravity, as it will be also explored in the Part II of the current thesis.

Quintessence

Of this models, a special mention is required for quintessence, which is probably the most known scalar field candidate for dark energy. First proposed by Caldwell *et al* in Ref. [71], it has been well studied during the past ten years. Quintessence can be described by the action (1.5) with $f(\phi) = 1$, where the kinetic term $\omega(\phi) > 0$ is canonical, and with a specific potential, whose choice will restrict the behavior of the scalar field, and then, the cosmological evolution. For a suitable choice of the scalar potential, late-time acceleration can be well reproduced. This kind of model will be studied in Part 1, where the possibility to reproduce not just dark energy, but also inflation through the same scalar field is explored. However, let us now study some qualitative properties of this kind of models. By writing the scalar field as a perfect fluid, its effective EoS yields,

$$w_\phi = \frac{p_\phi}{\rho_\phi} = \frac{\dot{\phi}^2 - 2V(\phi)}{\dot{\phi}^2 + 2V(\phi)} = -1 + \frac{2\dot{\phi}^2}{\dot{\phi}^2 + 2V(\phi)} \,. \tag{1.6}$$

here for simplicity we have taken $\omega(\phi) = 1$, what just corresponds to a redefinition of the scalar field. On the other hand, the second derivative of the scale factor can be written as,

$$\frac{\ddot{a}}{a} \propto -(\rho + 3p) \,, \tag{1.7}$$

In order to have an accelerating expansion for a perfect fluid with $p = w\rho$, it is necessary to have an EoS parameter $w < -1/3$. Then, for the case of a quintessence field, looking at the EoS (1.7), we see that $V(\phi) > \dot{\phi}^2$ has to be satisfied in order to have cosmic acceleration. Otherwise, if $V(\phi) << \dot{\phi}^2$, the EoS parameter gives $w_\phi \sim 1$, which corresponds to a contracting Universe. In the case of slow-roll inflation, we have that the slow-roll limit imposes $V(\phi) > \dot{\phi}^2$, which yields $w_\phi \sim -1$, that is basically an effective cosmological constant.

In part 1, it will be studied how this kind of fields could be used to reconstruct, not only dark energy or inflation separately, but both events unified under the same scalar field. This unification can well be achieved, as it will be shown, by choosing an appropriate kinetic and potential terms for the scalar field. Then, canonical scalar fields can be used to reproduce, at least in an effective way, the entire history of the Universe.

Phantom

Phantom fluids, first named by Caldwell in Ref. [70] can be easily described by scalar fields. For the action (1.5), we can take a minimally coupled field, $f(\phi) = 1$, where its kinetic term $\omega(\phi)$ becomes negative. This kind of fields yields an EoS parameter,

$$w_\phi < -1 \ . \tag{1.8}$$

The kind of fluids with such EoS parameter has several consequences both at the cosmological level and in microphysics. Among the strong energy condition, it violates also the weak one, while a cosmological constant or a quitessence field only violate the strong condition. The phantom field described by the action (1.5) owns a negative energy, what can lead to imaginary effective masses. From a microscopical point of view, it is well known that phantom fluids own serious problems, where its presence could introduce catastrophic quantum instabilities of the vacuum energy (see Refs. [70, 95]). However, one could study such kind of fluids dealing with them in an effective description, just valid to some energy scale, or where they are just a consequence of extra geometrical terms in the action (as it will be explored). From a cosmological point of view, we have that a phantom fluid produces a super-acceleration of the expansion that ends in a singularity, called Big Rip[7]. Let us consider a simple phantom model to show its consequences, by considering a constant EoS paremeter less than -1. Then, a general solution for a flat FLRW metric yields,

$$H(t) \propto \frac{1}{t_s - t} \ , \quad \rho_\phi \propto \frac{1}{(t_s - t)^2} \tag{1.9}$$

where $H = \frac{\dot{a}}{a}$ is the Hubble parameter, and t_s is the so-called Rip time, defined as the time when the future singularity occurs. Then, for $t \to t_s$, the Hubble parameter diverges producing a curvature singularity. Note that as the Universe evolves, the acceleration is increased and the phantom energy density becomes larger, such that sufficiently close to the Big Rip singularity, the strength of the phantom would be so large that breaks local systems as galaxies or solar systems, and even molecules and atoms. However, close to the singularity quantum effects should become important, and could affect the occurrence of the singularity, an aspect studied in the last chapters of the thesis, where semiclassical effects are taken into account.

Nevertheless, one could ask why not to restrict our cosmological models to EoS parameters greater than -1, such that phantom fluids would be avoided, but observations have pointed to the real possibility that dark energy behaves as a phantom fluid, what makes them of enough importance to be studied.

Non-minimally scalar-tensor theories

In this kind of theories, the scalar field is strongly coupled to the metric field through the Ricci scalar in the action. Then, for the action (1.5) we would have $f(\phi)$ not a constant, but a function of the field ϕ. The first proposed model of non-minimally coupled scalar-tensor theory was constructed by Brans and Dicke in 1961 [42], which was the first real attempt of an alternative to General Relativity with the main objective to incorporate Mach's principle into the theory of gravity, which is not satisfied in GR as it was well known already at that time. The model proposed by Brans and Dicke was described by an action of the same type as (1.5), where $f(\phi) = \phi$, $\omega(\phi) = \frac{\omega}{\phi}$ and with a null potential $V(\phi) = 0$. In spite of that is a theory that presents various problems, as corrections to the Newtonian potential or violation of the Birkhoff's theorem, it gave an alternative theory that provided a way to test GR in the experiments. Nowadays, these theories have recovered some interest due to their cosmological implications, where dark

[7] Note that phantom models free of singularities can be reconstructed. This aspect will be explored along the thesis.

energy can be well reproduced. Also it can be a very useful instrument to analyze modified gravity theories as $f(R)$ gravity due to the mathematical equivalence between them, as it will be studied in the thesis.

However, theories with a non-minimally scalar field could violate equivalence principle, which is very well tested at least at small scales (Earth, solar system, galaxies...). It is well known that in the Jordan frame, on which the action (1.5) is defined, a test particle will follow a geodesic, while in the Einstein frame (related to the Jordan's one by a conformal transformation), the path of a test particle moving in a given spacetime will suffer of corrections on its geodesic due to the appearance of a "fifth force", produced by the scalar field[8]. In Brans-Dicke theory, this violation in the Einstein frame can be written by the geodesic equation of a test particle, what yields a fifth force correction,

$$\frac{d^2x^\mu}{d\tau^2}+\Gamma^\mu_{\nu\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau}=-\beta\partial^\mu\phi\ , \tag{1.10}$$

where β is a free parameter of the theory related to ω in (1.5). Then, assuming that Einstein's frame is physical, this fifth force can be checked by the experiments. In 2003, the Casini spacecraft restricted the value of the parameter to be $\beta<1.6\times 10^{-3}$ (see Ref.[30]). However, another solution to the problem of fifth forces and corrections to the Newtonian potential, is the so-called "chameleon mechanism", proposed in Ref. [183], where the scalar field depends on the scale, such that it becomes important at large scales, where dark energy is needed, and it is neglible at local scales, satisfying the gravitational experiments. Here, the models will be analyzed where the coupling is given by $f(\phi)=1+g(\phi)$, being $g(\phi)$ a function that can be restricted at small scales by the chameleon mechanism, where GR is recovered in the action (1.5). It will be shown how this kind of theories can well reproduce late-time acceleration, as well as its relation with the results obtained in the Jordan and Einstein frame, and its possible physical implications. It will be also used as an auxiliary field to reconstruct modified theories of gravity able to reproduce cosmic acceleration.

1.3 Modifying General Relativity: towards a complete theory of gravity

Along the present thesis, the analysis on Extended Theories of Gravity (ETG) is one of main points to be studied. In this brief introduction, it will be argued the motivations for an extension of General Relativity to more complex and also general, laws of gravity. GR has been along the last one hundred years a very successful theory in order to explain the macroscopic world, on this sense GR gives a good description of nature at scales of the Earth or the Solar system, as the experiments have well proved. However, as it was commented above, GR presents some shortcomings when it is tested at large scales, these problems can be interpreted as unknown forms of matter/energy or as a failure of the theory. It has been well known since long time ago, that rotational curves of galaxies can not be fitted in GR unless dark matter or some modification is taken into account, as well as at cosmological scales it is necessary to incorporate a perfect fluid with negative pressure to explain late-time acceleration. Then, we have to decide if dark matter and/or dark energy could be a sufficiently satisfactory explanation or if our theory of gravity actually presents an scale of validity, and has to be modified beyond some limit. There is not a right answer to this serious dilemma yet, or at least a consensus over the scientific community to decide for one or another option. Nevertheless, there is an historical analogy to this current problem that could serve as an example of how to proceed: at the end of the nineteenth century, it was believed (with almost no doubt) that a fluid called aether existed, and everything was immersed on it, light or the planets were moving through

[8]Note that the physical equivalence between the Jordan and the Einstein frame is not completely clear, and big discussions have been going on this topic with no successful answer yet. For a review on it see Ref. [146] and references therein. For recent results, see Ref. [80, 85].

it, and it served as the inertial frame to refer the movements of any test body. The end of this history is well known by everyone. In the same way, dark energy and specially dark matter, have been searched over the last years with no result, what has opened the real possibility to investigate theories beyond GR in order to explain such phenomena[9]. It will be shown that generalizing Einstein's field equations, any cosmological solution can be reproduced, and then, late-time acceleration and even its unification with inflation can well be explained in the context of ETGs.

However, as it was pointed in the above section and it will be shown in the first part at the present thesis, scalar fields can easily reproduce the cosmological evolution, and do not introduce so big changes as modified gravity does (except non-minimally scalar-tensor theories, which from a mathematical point of view are equivalents to some kind of modified gravities). However, as it has been pointed, quintessence or phantom fields are exotic forms of matter/energy never observed directly, and with an unknown origin. On the other hand, modified gravities offer an alternative explanation, which seems to be more natural and could avoid the coincidence problem unifying inflation and late-time acceleration under the same mechanism. Also, it has been recently shown that some "viable" modified gravities could keep GR at small scales almost unchanged, avoiding the terrible problems of violation of local tests of gravity that usually affect to ETGs. However, one has to say that modified gravities have to be seen not as the ultimate theory of gravity that resolves all the theoretical and observational problems, but as an effective description of a probably more complex theory.

Let us deeper analyze the statements and motivations to consider GR as a good description of nature at some scale, but which fails in some limits, where corrections are needed. The principles on which General Relativity is based, and which are well tested at least at macroscopic local scales, have to be satisfied by any extension of GR on those scales. General Relativity is based on three assumptions[10]:

- The Principle of Relativity. The laws of nature are the same for every observer, what means that there is not preferred system of reference.
- The Principle of Equivalence, which establishes the equivalence between accelerated observers and those immersed in a gravitational field. In a sample form, it imposed the equivalence between the inertial and gravitational masses.
- The Principle of General Covariance. It implies that every law's equation of nature has to be covariant, what means invariant under general transformation of coordinates.

However, these assumptions tell us little about the form of the field equations. In order to restrict our theory of gravity, we can impose two additional conditions, that will constraint the field equations,

- The energy has to be conserved, what implies that the action of the theory has to be invariant under general transformations of coordinates.
- At the non-relativistic limit, Newtonian law has to be recovered, or basically, at some limit, Special Relativity has to be recovered.

It is interesting to point that the first proposal for gravitational field equations by Einstein was in 1913, and it did not satisfy the above conditions. He proposed probably the most simple construction using the

[9]Some modified gravities try to explain the origin of dark matter as a pure gravitational effect. Several attempts have been performed on this sense, as MONDs or even $f(R)$ gravity. At the present work, we are not focusing on this aspect, but on dark energy and inflation.

[10]For a complete analysis on General Relativity see for example the textbooks Refs. [299, 303]

Ricci tensor with the following equations for gravity,

$$R_{\mu\nu} = \kappa^2 T_{\mu\nu} \ , \tag{1.11}$$

where $R_{\mu\nu}$ is the Ricci tensor, $T_{\mu\nu}$ is the energy-momentum tensor and κ^2 the coupling constant. It is easy to see that the field equations (1.11) do not satisfy the energy conservation as the covariant derivative of the equation is not null, $D_\mu R^{\mu\nu} \neq 0$, for a general spacetime. After this first attempt, Einstein arrived to the well-known field equations for GR,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa^2 T_{\mu\nu} \ , \tag{1.12}$$

which now satisfies $D_\mu G^{\mu\nu} = 0$. Also on the Newtonian limit of the theory, the field equations (1.12) reduce to the Newtonian law in a non-relativistic limit. It is straightforward to show that for a weak and static limit, the field equation (1.12) yields to,

$$G_{\mu\nu} \sim \nabla^2 g_{00} = -\kappa^2 T_{00} \ , \tag{1.13}$$

which coincides with the Newtonian law. Then, one can impose the above two conditions in order to construct a realistic theory of gravity. Instead of the field equations (1.12), GR can be expressed in a more elegant way through the action principle,

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} R \ . \tag{1.14}$$

This action, called the Hilbert-Einstein action, is invariant under general transformation of coordinates, what yields to conserved field equations. Then, one could propose an invariant action, more general than (1.14), but which could evolve more complex invariants as the terms $R^2, R^{\mu\nu}R_{\mu\nu}, R^{\sigma\lambda\mu\nu}R_{\sigma\lambda\mu\nu}, f(G)...$. However, these additional terms could imply corrections in the Newtonian law (see Ref. [287]), although can be neglected. Then, the kind of theories studied here will be of the form,

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} f(R, G) \ . \tag{1.15}$$

Here $f(R, G)$ is an arbitrary function of the Ricci scalar and the Gauss-Bonnet term, the arbitrariness conveys the ignorance of our knowledge on the right theory of gravity, but which can be constrained to satisfy the requirements of the observational data among the requirements written above, and could provide an explanation for the cosmic acceleration with no additional fields (see Ref. [74]). The action (1.15) is invariant under general transformation of coordinates but yields corrections on the Newtonian law, which can be controlled by specific functions and parameters of the theory (see for example Ref. [86]). However, in this case the field equations are fourth order instead of second order in GR, what implies more complexity on the equations, a not well formulated Cauchy problem unless some conditions are imposed (see Ref. [78]) or massive gravitons. However, these problems can be resolved by restricting the action(1.15) and the free parameters of the theory, what constraints the gravity law.

From a quantum point of view, there is not a successful and consistent theory of quantum gravity yet, i.e., a renormalizable and unitary theory. However, much before higher order theories of gravity became of interest for low energy scales (late cosmology), they were of great interest in order to construct a quantum theory of gravity. It is well known that GR is a non-renormalizable theory, but when extra geometrical terms are added in the action, the theory turns to be renormalizable at one-loop, but it fails at two-loops. It has been shown that when quantum corrections are taken, the low effective action admits higher order terms (see Ref. [63]). Then, it is natural to think that gravity admits extra terms in the action coming from quantum corrections. However, recently a new proposal on quantum gravity, performed by Hořava [174], claims to be power counting renormalizable, and where instead to new terms on the action, it imposes

non-covariant equations. This new theory has became very popular as it presents a new path on the way to find a consistent theory of quantum gravity. This theory will be discussed in the next section, but among serious problems that it has, it suffers of the same cosmological problems as GR does.

Mathematically, ETGs provide very interesting properties to be analyzed. Among the theories of higher order which are intrinsically interesting to study, modified gravities introduces new ways to study gravitational physics. On this sense, in $F(R)$ theories, one can proceed in two ways, called the metric and Palatini formalism respectively. The metric approach assumed that the connection $\Gamma^{\lambda}_{\mu\nu}$ depends and is defined in terms of the metric field $g_{\mu\nu}$, which is the only field of the theory. While in the Palatini's approach, one does not assume that the connection $\Gamma^{\lambda}_{\mu\nu}$ is already defined but it is dealt with a dynamical field as the metric tensor. In GR, both approaches are equivalent, as in Palatini the connection naturally appears defined in terms of $g_{\mu\nu}$, but in higher order theories they are not equivalent, and one has to distinguish between both approaches. Along the present thesis, it will be analyzed only the metric approach. For a review on Palatini's $F(R)$ gravities see Ref. [281] and references therein.

As we will see, the action (1.15) can be chosen such that, among the problems commented, it can reproduce late-time acceleration and even inflation together, with no need of an extra exotic fluid. As the extra geometrical terms in the action can reproduce cosmic acceleration, what basically means that behaves as an effective perfect fluid with an EoS parameter $w \sim -1$ at the current epoch, it could produce a relaxation on the value of the vacuum energy density, resolving the cc problem by a simple mechanical way (see Ref. [26]). Some gravity theories will be studied, where the action is given by the one of Hilbert-Einstein action (1.14) plus a function of the Ricci and/or Gauss-Bonnet scalars that would become important at large scales but reduces to GR for local scales. Note that for higher order gravities as (1.15), the presence of various de Sitter solutions in the equations can provide a way to unify inflation and late-time acceleration under purely gravitational terms, and even it could predict future phases of accelerated expansion, what avoids the coincidence problem. Also it is important to point that, in general, the extra terms in the gravitational action can be modeled as a perfect fluid, whose effective EoS parameter will not be a constant in general, and it could vary along the Universe evolution, even crossing the phantom barrier, providing a way to interpret EoS parameters less than -1. As has been pointed out, these theories are mathematically equivalent to non-minimally scalar-tensor theories, what can be very useful for the analysis and reconstruction of the action for modified gravity. Also the relation between Einstein and Jordan frames provide a way to reconstruct modified gravities from scalar field models, in spite of the fact that cosmological solutions will not coincide in both frames, what could give an explanation of the (non-)physical equivalence between them. Moreover, it is well known that phantom scalar fields have not a representation in the Jordan frame as the scalar kinetic term is negative, what implies a complex conformal transformation that yields an action that becomes complex in the Jordan frame [60]. This problem will be analyzed, and this kind of theories will be studied with the presence of fluids with different EoS.

Hence, it will be here shown that theories of gravity can explain the cosmological shortcomings, and at the same time, remain unchanged the principles of GR at local scales, where the gravitational test on GR are well known. Also, the reconstruction of any kind of cosmological solution will be analyzed, including Λ CDM model, which gives interesting properties in modified gravity. this kind of theories with the presence of fluids with different EoS, and its cosmological evolution, will be studied.

Hořava-Lifshitz gravity

In this introduction a special mention is due to the so-called Hořava-Lifshitz (HL) gravity for it is a new theory that has risen to a lot of interest in the scientific community. This new theory, proposed just two years ago by Hořava Ref. [174] claims to be power counting renormalizable by losing the covariance of the theory. It is well known that GR is not renormalizable, the main problem in perturbative renormalization

comes from the fact that the gravitational coupling G is dimensionful, with a negative dimension $[G] = -2$ in mass units, what gives uncontrolled UV divergences. This can be cured by introducing higher terms of the scalar curvature in the action, but at the same time it provides an additional problem on the uniqueness. The main success of HL theory is that avoids these problems, and gives a power counting renormalizable theory by introducing an anisotropic scaling property between space and time characterized by a dynamical critical exponent z. This kind of anisotropy is common in condensed matter physics, introduced firstly by Lifshitz [200] to study some kind of phase transitions, it is imposed a different degree between the spatial and time components. In HL gravity, this scaling property can be written as,

$$x^i = bx^i\,, \quad t = b^z t\,. \tag{1.16}$$

The theory becomes power counting renormalizable in 3+1 spacetime dimension when the critical exponent is $z = 3$, while GR is recovered when $z = 1$. This scaling property makes the theory to not be invariant under diffeomorphism but under some restricted transformations called foliation-preserving diffeomorphisms,

$$\delta x^i = \zeta(x^i, t)\,, \quad \delta t = f(t)\,. \tag{1.17}$$

Then, the theory breaks Lorentz invariance, what would have very serious consequences as different observers would measure different values of the speed of light c. However, it is assumed that the full diffeomorphism is recovered in some limit, where the critical exponent flows to $z = 1$ and the symmetries of GR are recovered, although the mechanism for this transition is not clear yet.

Respect the action of the theory, let us write the scalar Ricci in the ADM decomposition [14],

$$R = K_{ij}K^{ij} - K^2 + R^{(3)} + 2\nabla_\mu(n^\mu\nabla_\nu n^\nu - n^\nu\nabla_\nu n^\mu)\,, \tag{1.18}$$

here $K = g^{ij}K_{ij}$, K_{ij} is the extrinsic curvature, $R^{(3)}$ is the spatial scalar curvature, and n^μ a unit vector perpendicular to a hypersurface of constant time. The action proposed in [174] accounts the anisotropic scaling properties (1.16) by introducing additional parameters (μ, λ) in the theory,

$$S = \frac{1}{2\kappa^2}\int dt d^3x\sqrt{g^{(3)}}N\tilde{R}\,, \quad \tilde{R} = K_{ij}K^{ij} - \lambda K^2 + R^{(3)} + 2\mu\nabla_\mu(n^\mu\nabla_\nu n^\nu - n^\nu\nabla_\nu n^\mu) - L^{(3)}(g_{ij}^{(3)})\,. \tag{1.19}$$

Note that the term in front of μ can be dropped out as it turns out to be a total derivative, although it becomes important for extensions of the action (1.19). Then, the term $L^{(3)}(g_{ij}^{(3)})$ is chosen to be the variation of an action in order to satisfy an additional symmetry, the detailed balanced, introduced by Hořava in Ref. [174]. However, from a cosmological point of view, HL theory suffers from the same problems as GR, additional components are needed to explain late-time acceleration or inflation. Then, following the same procedure as in GR, the HL action can be extended to more general actions [100],

$$S = \frac{1}{2\kappa^2}\int dt d^3x\sqrt{g^{(3)}}NF(\tilde{R})\,. \tag{1.20}$$

It will be shown that this kind of action can well reproduce the cosmological history and avoid corrections to the Newtonian law.[11] Even how the matter/radiation epochs of the cosmic evolution yields to an accelerated epoch modeled by the kind of theories described by (1.19).

Nevertheless, the theory proposed by Hořava, or its extension, described by the action (1.20) contain several problems that makes the theory to not be a good consideration. The lose of symmetries due to the anisotropic scaling (1.16) introduces an additional spin-0 mode that can rise some problems. In

[11] Here we do not discuss the gravity sector corresponding to the Hořava gravity but we show that the scalar mode, which also appears in the usual $F(R)$ gravity, can decouple from gravity and matter, and then the scalar mode does not give a measurable correction to Newton's law.

order to satisfy the observations, such scalar mode has to decouple in the IR limit, where GR has to be recovered. However, this is not the case here, where the scalar mode introduces strong instabilities around flat spacetime. One possible solution for this trouble is to consider an stable de Sitter spacetime as the natural vacuum solution of the theory, what can be achieved by some specific actions $F(\tilde{R})$. Another solution, proposed in Ref. [175], is to include an additional local $U(1)$ symmetry, what could force to the additional parameters (λ, μ) to be equal to 1, neglecting the spin-0 mode in the IR limit. The extension of this kind of HL gravity will be studied here.

Hence, the new proposal as candidate for quantum field theory of gravity seems to be a good promise, in spite of the serious problems that owns itself. Here, their applications to cosmology will be explored, and the possible solutions to the intrinsic problems of the theory as well as the possibility to reproduce inflation and late-time acceleration under the theory with no additional components.

1.4 Future singularities in FLRW universes

A wide range of solutions in GR or ETGs contain singularities. Some examples can be found in spherical symmetric solutions in vacuum as Schwarzschild or Kerr metrics, which contain a curvature singularity at the origin of the radial coordinate, or the FLRW metric, where in general contains an initial singularity, the so-called Big Bang. The singularities can be viewed as the limits of our law of gravity, where a consistent theory of quantum gravity or a theory of “everything“ as String theory is needed. However, the singularities can be treated in a semiclassical approach introducing some corrections coming from quantum field theory in the field equations, to see if some corrections could cure the singularity. The last part of the present thesis is devoted to this subject, (future) singularities of FLRW metrics will be analyzed when some effects are taken into account as the effective cosmological Casimir effect or the conformal anomaly. Also, the validity of holographic principle will be studied, where a bound on the Universe entropy is imposed, whose validity around future singularities is studied and interesting results are obtained. In parallel, it is explored the possibility of a generalization of Cardy-Verlinde (CV) formula Ref [297]. CV formula suggests a more fundamental origin for the Friedman equation, which coincides with a two-dimensional conformal field theory for a closed FLRW metric with some particular requirement when a bound on the entropy proposed by Verlinde Ref [297] is reached.

Let us define the list of possible singularities contained in FLRW universes. It has been pointed out that, apart Big Bang initial singularity, some kind of perfect fluids, specially phantom fluids, could drive the Universe to future singularities of different type. In Ref. [246], following classification for future singularities was obtained,[12]:

- Type I (“Big Rip”): For $t \to t_s$, $a \to \infty$ and $\rho \to \infty$, $|p| \to \infty$.
- Type II (“Sudden”): For $t \to t_s$, $a \to a_s$ and $\rho \to \rho_s$, $|p| \to \infty$. (see Refs.[23, 236])
- Type III: For $t \to t_s$, $a \to a_s$ and $\rho \to \infty$, $|p| \to \infty$.
- Type IV: For $t \to t_s$, $a \to a_s$ and $\rho \to \rho_s$, $p \to p_s$ but higher derivatives of Hubble parameter diverge.

Here ρ and p are referred to the energy and pressure densities of the fluid responsible for the occurrence of the future singularity. The most usual future singularity in FLRW metrics is the first type of the above

[12]Note that we have omitted here other kind of future singularities as the so-called ”Big Crunch” as we are interested to study the future state of an accelerating Universe and not a contracting one.

list, the so-called Big Rip singularity, which is usually produced by a phantom fluid whose EoS parameter $w < -1$, and whose origin may be an scalar field or pure geometrical additional terms in the action. As the singularities are regions or points of a spacetime, where the classical theory fails, it is natural to introduce some quantum corrections in the field equations. Here, the consequences of considering an effective Casimir effect will be analyzed (dealing the Universe as an spherical shell) near the singularity, on one side, and we will account for the effects of the effective action produced by conformal anomaly, on the other side. It will be shown that for a general case, those effects incorporated in the theory do not cure the singularity, such that additional semiclassical terms should be taken into account. Also, it will be shown that "universal" bounds of the entropy are violated near and even far away from the singularity, losing its universal nature.

On the other hand, the construction of ETGs free of future singularities is also studied. It is well known that the majority of higher order gravities hide solutions for a FLRW Universe that contain some class of the above future singularities. It was found that an action with $F(R) = R^2$ is free of singularities and even any standard modified gravity where a term proportional to R^2 is added, can cure the singularity (see Refs. [1, 239]). This analysis is extended to the Hořava-Lifshitz gravity, where the terms that can avoid the occurrence of the future singularity are found, too.

1.5 Organization of the thesis

The thesis is composed of three main blocks, each of them dedicated to cosmological models and aspects of FLRW metrics. As it has been pointed out in this brief introduction, the main purpose of the present thesis is the reconstruction of cosmic evolution by several different ways, but where modified gravities have an special interest. In parallel to this main objective, other related topics are studied, which have something to do with FLRW Universes. On those instances where explicitly specified, we will deal with spatially flat FLRW metrics.

In the first block, scalar-tensor theories are analyzed; there quintessence/phantom scalar fields as well as Brans-Dicke-like theories are studied. Unification of inflation and late-time acceleration can be easily achieved on this context, even the problem on the transition to a phantom epoch (big instability) can be resolved by not allowing a transition of the scalar field, in this way the observational current data can be better fitted using more than one scalar field. It is also explored in the context of non-minimally scalar-tensor theories the possibility to reproduce the Universe evolution and the related solutions in Jordan and Einstein frames, where it is shown that both solutions are in general very different, and for example the occurrence of a singularity in one frame can be avoided in the other, what could be a signal of the non-physical equivalence between both frames. In this first block, it is also explored the possibility of an oscillating Universe, where the accelerated phases of the Universe would be repeated until a perturbation or a future singularity ends with this periodic behavior of the Universe. It is shown that such oscillations can be reproduced easily by an scalar field. This could resolve the so-called coincidence problem, where the period of the oscillations can be fixed with the age of the Universe. Also, the possibility to have an interacting dark energy is studied, which exchanges energy with dust matter. And how the interacting term could yield an accelerated expansion.

The second block is dedicated to the analysis of Extended Theories of Gravity, specially extensions of General Relativity where more general functions of the Ricci and/or Gauss-Bonnet scalars are involved in the action. It is shown through out the equivalence of $f(R)$ gravities and non-minimally scalar-tensor theories that this kind of modified gravity admits almost any class of solutions for a flat FLRW Universe. It is also shown that $f(R)$ gravity can be dealt with as an effective perfect fluid with a dynamical EoS. Some phantom cosmological models free of future singularities are constructed on this frame. It is shown that

unification of inflation and late-time acceleration can easily be achieved in these theories, what provides a natural explanation for accelerating expansion phases of the Universe. A reconstruction method for cosmological solutions is also implemented, where no auxiliary field are used. In this case, Λ CDM model is constructed in modified gravities with no cosmological constant presence in the action, and the solution is a series of powers of the Ricci scalar. In the context of viable modified gravities, the ones that have a good behavior at local scales, the cosmological evolutions in comparison with those of the Λ CDM model are analyzed, as well as in the case when an additional phantom fluid is taken into account, the scalar-tensor representation for this class of theories is obtained and analyzed. Finally, as in $f(R)$ theories, models are studied where the modification of GR comes from functions of the Gauss-Bonnet invariant G, where it is shown that the use of a combination of Ricci/Gauss-Bonnet terms provide a theory of gravity.

The recent proposal of quantum gravity, Hořava-Lifshitz gravity, is explored in the third block. This theory is power-counting renormalizable at the price of breaking Lorentz invariance. Apart from some intrinsic problems of the theory, as its transition to recover the full diffeomorphisms, HL gravity suffers problems of the same kind as GR when FLRW metrics are considered. On this sense, an exotic fluid or a cc is needed to explain cosmic acceleration. Extensions of the action proposed by Hořava are studied, where the Newtonian limit is studied. It is also found the condition to have a theory free of future singularities. The stability of the cosmic solutions is explored, where some general conditions on the action for HL gravity are obtained. Several examples are analyzed to illustrate it. By following these stability conditions, one can reconstruct the entire Universe history. Finally, one of the intrinsic problems of the theory is analyzed, the scalar mode associated to the lose of full diffeomorphism that affects the vacuum solution of the theory. The solution to this problem can come from adding an additional U(1) symmetry or from fixing the vacuum solution as de Sitter. Both possibilities are explored.

Finally, in the last part of the thesis, other aspects of FLRW metrics are studied. Generalizations of the Cardy-Verlinde formula, where different types of fluids are involved, are explored. It is shown that only for special cases, the original CV formula is recovered. Also dynamical entropy bounds are analyzed close to future singularities, where some semiclassical effects are taken into account. The inclusion of an effective Casimir effect is studied close to the singularity and even along the Universe evolution, its effects are studied.

The thesis ends with the general conclusions corresponding to the results obtained in the present work.

Part I

Dark energy and inflation from scalar-tensor theories and inhomogeneous fluids

Chapter 2

Reconstructing inflation and cosmic acceleration with phantom and canonical scalar fields

[1]The increasing amount and precision of observational data demand that theoretical cosmological models be as realistic as possible in their description of the evolution of our universe. The discovery of late-time cosmic acceleration brought into this playground a good number of dark energy models which aim at describing the observed accelerated expansion, which seems to have started quite recently on the redshift scale. Keeping in mind the possibility, which is well compatible with the observational data, that the effective equation of state parameter w be less than -1 , phantom cosmological models [2, 9, 8, 13, 15, 22, 24, 60, 62, 67, 72, 102, 103, 104, 114, 115, 133, 137, 140, 141, 143, 153, 159, 164, 167, 171, 180, 181, 202, 203, 215, 220, 224, 225, 246, 255, 271, 274, 275, 278, 279, 284, 288, 292, 294, 298, 302, 306, 309, 310, 311] share a place in the list of theories capable to explain dark energy. Ideal fluid and scalar field quintessence/phantom models still remain among the easiest and most popular constructions. Nevertheless, when working with these models, one should bear in mind that such theories are at best effective descriptions of the early/late universe, owing to a number of well-known problems.

Even in such a situation, scalar field models still remain quite popular candidates for dark energy. An additional problem with these theories–which traditionally has not been discussed sufficiently–is that a good mathematical theory must not be limited to the description of a single side of the cosmic evolution: it should rather provide a unified description of the whole expansion history of the universe, from the inflationary epoch to the onset of cosmic acceleration, and beyond. Note that a similar drawback is also typical of inflationary models, most of which have problems with ending inflation and also fail to describe realistically the late-time universe.

The purpose of this chapter is to show that, given a certain scale factor (or Hubble parameter) for the universe expansion history, one can in fact reconstruct it from a specific scalar field theory. Using multiple scalars, the reconstruction becomes easier due to the extra freedom brought by the arbitrariness in the scalar field potentials and kinetic factors. However, there are subtleties in these cases that can be used advantageously, and this makes the study of those models even more interesting.

[1]This Chapter is based on the publications: [136, 264]

Specifically, in this chapter we overview the reconstruction technique for scalar theories with one, two, and an arbitrary number n of fields, as well as we consider a generalization of a type of Brans-Dicke theory, where the scalar field appears non-minimally coupled to the gravitational field, what could produce violations of gravitational tests at local scales. In order to satisfy the observational constraints the so-called chameleon mechanism is studied (see Ref. [183]). Many explicit examples are presented in which a unified, continuous description of the inflationary era and of the late-time cosmic acceleration epoch is obtained in a rather simple and natural way.

2.1 Unified inflation and late time acceleration in scalar theory

Let us consider a universe filled with matter with equation of state $p_m = w_m \rho_m$ (here w_m is a constant) and a scalar field which only depends on time. We will show that it is possible to obtain both inflation and accelerated expansion at late times by using a single scalar field ϕ (see also [82] and [231]). In this case, the action is

$$S = \int dx^4 \sqrt{-g} \left[\frac{1}{2\kappa^2} R - \frac{1}{2}\omega(\phi)\partial_\mu\phi\partial^\mu\phi - V(\phi) + L_m \right] , \tag{2.1}$$

where $\kappa^2 = 8\pi G$, $V(\phi)$ being the scalar potential and $\omega(\phi)$ the kinetic function, respectively, while L_m is the matter Lagrangian density. Note that for convenience the kinetic factor is introduced. At the final step of calculations, scalar field maybe always redefined so that kinetic factor is absorbed in its definition. As we work in a spatially flat Friedmann-Robertson-Walker (FLRW) spacetime, the metric is given by

$$ds^2 = -dt^2 + a^2(t) \sum_{i=1}^{3} dx_i^2 \ . \tag{2.2}$$

The corresponding FLRW equations are written as

$$H^2 = \frac{\kappa^2}{3} \left(\rho_m + \rho_\phi\right) \ , \qquad \dot{H} = -\frac{\kappa^2}{2} \left(\rho_m + p_m + \rho_\phi + p_\phi\right) \ , \tag{2.3}$$

with ρ_ϕ and p_ϕ given by

$$\rho_\phi = \frac{1}{2}\omega(\phi)\,\dot{\phi}^2 + V(\phi) \ , \qquad p_\phi = \frac{1}{2}\omega(\phi)\,\dot{\phi}^2 - V(\phi) \ . \tag{2.4}$$

Combining Eqs. (2.3) and (2.4), one obtains

$$\omega(\phi)\,\dot{\phi}^2 = -\frac{2}{\kappa^2}\dot{H} - (\rho_m + p_m) \ , \qquad V(\phi) = \frac{1}{\kappa^2}\left(3H^2 + \dot{H}\right) - \frac{\rho_m - p_m}{2} \ . \tag{2.5}$$

As the matter is not coupled to the scalar field, by using energy conservation one has

$$\dot{\rho_m} + 3H(\rho_m + p_m) = 0 \ , \quad \dot{\rho_\phi} + 3H(\rho_\phi + p_\phi) = 0 \ . \tag{2.6}$$

From the first equation, we get $\rho_m = \rho_{m0} a^{-3(1+w_m)}$. We now consider the theory in which $V(\phi)$ and $\omega(\phi)$ are

$$\omega(\phi) = -\frac{2}{\kappa^2} f'(\phi) - (w_m + 1) F_0 \mathrm{e}^{-3(1+w_m)F(\phi)} \ , \quad V(\phi) = \frac{1}{\kappa^2}\left[3f(\phi)^2 + f'(\phi)\right] + \frac{w_m - 1}{2} F_0\,\mathrm{e}^{-3(1+w_m)F(\phi)} \ , \tag{2.7}$$

where $f(\phi) \equiv F'(\phi)$, F is an arbitrary (but twice differentiable) function of ϕ , and F_0 is an integration constant. Then, the following solution is found (see [75, 82, 231, 247]):

$$\phi = t \ , \quad H(t) = f(t) \ , \tag{2.8}$$

which leads to

$$a(t) = a_0 \mathrm{e}^{F(t)}, \qquad a_0 = \left(\frac{\rho_{m0}}{F_0}\right)^{\frac{1}{3(1+w_m)}} . \tag{2.9}$$

We can study this system by analyzing the effective EoS parameter which, using the FLRW equations, is defined as

$$w_{\text{eff}} \equiv \frac{p}{\rho} = -1 - \frac{2\dot{H}}{3H^2} , \tag{2.10}$$

where

$$\rho = \rho_m + \rho_\phi , \quad p = p_m + p_\phi . \tag{2.11}$$

Using the formulation above, one can present explicit examples of reconstruction as follows.

Example 1

As a first example, we consider the following model:

$$f(\phi) = h_0^2 \left(\frac{1}{t_0^2 - \phi^2} + \frac{1}{\phi^2 + t_1^2}\right) . \tag{2.12}$$

Using the solution (2.9), the Hubble parameter and the scale factor are given by

$$H = h_0^2 \left(\frac{1}{t_0^2 - t^2} + \frac{1}{t^2 + t_1^2}\right) , \quad a(t) = a_0 \left(\frac{t + t_0}{t_0 - t}\right)^{\frac{h_0^2}{2t_0}} e^{\frac{h_0^2}{t_1} \arctan \frac{t}{t_1}} . \tag{2.13}$$

As one can see, the scale factor vanishes at $t = -t_0$, so we can fix that point as corresponding to the creation of the universe. On the other hand, the kinetic function and the scalar potential are given by Eqs. (2.7), hence

$$\begin{aligned} \omega(\phi) &= -\frac{8}{\kappa^2} \frac{h_0^2(t_1^2 + t_0^2)\left(\phi^2 - \frac{t_1^2+t_0^2}{2}\right)\phi}{(t_1^2 + \phi^2)^2(t_0^2 - \phi^2)^2} - (w_m + 1)F_0 \mathrm{e}^{-3(w_m+1)F(\phi)} , \\ V(\phi) &= \frac{h_0^2(t_1^2 + t_0^2)}{\kappa^2(t_1^2 + \phi^2)^2(t_0^2 - \phi^2)^2} \left[3h_0^2(t_1^2 + t_0^2) + 4\phi\left(\phi^2 - \frac{t_1^2 + t_0^2}{2}\right)\right] + \frac{w_m - 1}{2} F_0 \; e^{-3(w_m+1)F(\phi)} \end{aligned} \tag{2.14}$$

where F_0 is an integration constant and

$$F(\phi) = \frac{h_0^2}{2t_0} \ln\left(\frac{\phi + t_0}{t_0 - \phi}\right) + \frac{h_0^2}{t_1} \arctan \frac{\phi}{t_1} . \tag{2.15}$$

Then, using Eq. (2.10), the effective EoS parameter is written as

$$w_{\text{eff}} = -1 - \frac{8}{3h_0^2} \frac{t(t - t_+)(t + t_-)}{(t_1^2 + t_0^2)^2} , \tag{2.16}$$

where $t_\pm = \pm\sqrt{\frac{t_0^2 - t_1^2}{2}}$. There are two phantom phases that occur when $t_- < t < 0$ and $t > t_+$, and another two non-phantom phases for $-t_0 < t < t_-$ and $0 < t < t_+$, during which $w_{\text{eff}} > -1$ (matter/radiation-dominated epochs). The first phantom phase can be interpreted as an inflationary epoch, and the second one as corresponding to the current accelerated expansion, which will end in a Big Rip singularity when $t = t_0$. Note that superacceleration (*i.e.*, $\dot{H} > 0$) is due to the negative sign of the kinetic function $\omega(\phi)$, as for "ordinary" phantom fields (to which one could reduce by redefining the scalar ϕ).

Example 2

As a second example, we consider the choice

$$f(\phi) = \frac{H_0}{t_s - \phi} + \frac{H_1}{\phi^2} \ . \tag{2.17}$$

We take H_0 and H_1 to be constants and t_s as the Rip time, as specified below. Using (2.7), we find that the kinetic function and the scalar potential are

$$\begin{aligned}
\omega(\phi) &= -\frac{2}{\kappa^2}\left[\frac{H_0}{(t_s-\phi)^2} - \frac{2H_1}{\phi^2}\right] - (w_m+1)F_0\,(t_s-\phi)^{3(1+w_m)H_0}\exp\left[\frac{3(1+w_m)H_1}{\phi}\right] \ , \\
V(\phi) &= \frac{1}{\kappa^2}\left[\frac{H_0(3H_0+1)}{(t_s-\phi)^2} + \frac{H_1}{\phi^3}\left(\frac{H_1}{\phi}-2\right)\right] + \frac{w_m-1}{2}F_0\,(t_s-\phi)^{3(1+w_m)H_0}\,\mathrm{e}^{\frac{3(1+w_m)H_1}{\phi}} \ ,
\end{aligned} \tag{2.18}$$

respectively. Then, through the solution (2.9), we obtain the Hubble parameter and the scale factor

$$H(t) = \frac{H_0}{t_s - t} + \frac{H_1}{t^2} \ , \quad a(t) = a_0\,(t_s - t)^{-H_0}\,\mathrm{e}^{-\frac{H_1}{t}} \ . \tag{2.19}$$

Since $a(t) \to 0^+$ for $t \to 0$, we can fix $t = 0$ as the beginning of the universe. On the other hand, at $t = t_s$ the universe reaches a Big Rip singularity, thus we keep $t < t_s$. In order to study the different stages that our model will pass through, we calculate the acceleration parameter and the first derivative of the Hubble parameter. They are

$$\dot{H} = \frac{H_0}{(t_s-t)^2} - \frac{2H_1}{t^3} \ , \quad \frac{\ddot{a}}{a} = H^2 + \dot{H} = \frac{H_0}{(t_s-t)^2}(H_0+1) + \frac{H_1}{t^2}\left(\frac{H_1}{t^2} - \frac{2H_1}{t} + \frac{2H_0}{t_s-t}\right) \ . \tag{2.20}$$

As we can observe, for t close to zero, $\ddot{a}/a > 0$, so that the universe is accelerated during some time. Although this is not a phantom epoch, since $\dot{H} < 0$, such stage can be interpreted as corresponding to the beginning of inflation. For $t > 1/2$ but $t \ll t_s$, the universe is in a decelerated epoch ($\ddot{a}/a < 0$). Finally, for t close to t_s, it turns out that $\dot{H} > 0$, and then the universe is superaccelerated, such acceleration being of phantom nature and ending in a Big Rip singularity at $t = t_s$.

Example 3

Our third example also exhibits unified inflation and late time acceleration, but in this case we avoid phantom phases and, therefore, Big Rip singularities. We consider the following model:

$$f(\phi) = H_0 + \frac{H_1}{\phi^n} \ , \tag{2.21}$$

where H_0 and $H_1 > 0$ are constants and n is a positive integer (also constant). The case $n = 1$ yields an initially decelerated universe and a late time acceleration phase. We concentrate on cases corresponding to $n > 1$ which gives, in general, three epochs: one of early acceleration (interpreted as inflation), a second decelerated phase and, finally, accelerated expansion at late times. In this model, the scalar potential and the kinetic parameter are given, upon use of Eqs. (2.7) and (2.21), by

$$\omega(\phi) = \frac{2}{\kappa^2}\frac{nH_1}{\phi^{n+1}} - (w_m+1)F_0\,\mathrm{e}^{-3(w_m+1)\left(H_0\phi - \frac{H_1}{(n-1)\phi^{n-1}}\right)}, \tag{2.22}$$

$$V(\phi) = \frac{1}{\kappa^2}\frac{3}{\phi^{n+1}}\left[\frac{\left(H_0\phi^{n/2}+H_1\right)^2}{\phi^{n-1}} - \frac{nH_1}{3}\right] + \frac{w_m-1}{2}F_0\,\mathrm{e}^{-3(w_m+1)\left(H_0\phi - \frac{H_1}{(n-1)\phi^{n-1}}\right)}. \tag{2.23}$$

Then, the Hubble parameter given by the solution (2.9) can be written as

$$H(t) = H_0 + \frac{H_1}{t^n} \ , \quad a(t) = a_0 \exp\left[H_0 t - \frac{H_1}{(n-1)t^{n-1}}\right] \ . \tag{2.24}$$

We can fix $t = 0$ as the beginning of the universe because at this point $a \to 0$, so $t > 0$. The effective EoS parameter (2.10) is

$$w_{\text{eff}} = -1 + \frac{2nH_1 t^{n-1}}{(H_0 t^n + H_1)^2} \ . \tag{2.25}$$

Thus, when $t \to 0$ then $w_{\text{eff}} \to -1$ and we have an acceleration epoch, while for $t \to \infty$, $w_{\text{eff}} \to -1$ which can be interpreted as late time acceleration. To find the phases of acceleration and deceleration for $t > 0$, we study $\ddot{a}/a$, given by:

$$\frac{\ddot{a}}{a} = \dot{H} + H^2 = -\frac{nH_1}{t^{n+1}} + \left(H_0 + \frac{H_1}{t^n}\right)^2 \ . \tag{2.26}$$

For sufficiently large values of n we can find two positive zeros of this function, which means two corresponding phase transitions. They happen, approximately, at

$$t_\pm \approx \left[\sqrt{nH_1}\ \frac{\left(1 \pm \sqrt{1 - \frac{4H_0}{n}}\right)}{2H_0}\right]^{2/n} \ , \tag{2.27}$$

so that, for $0 < t < t_-$, the universe is in an accelerated phase interpreted as an inflationary epoch; for $t_- < t < t_+$ it is in a decelerated phase (matter/radiation dominated); and, finally, for $t > t_+$ one obtains late time acceleration, which is in agreement with the current cosmic expansion.

Example 4

As our last example, we consider another model unifying early universe inflation and the accelerating expansion of the present universe. We may choose $f(\phi)$ as

$$f(\phi) = \frac{H_i + H_l c e^{2\alpha\phi}}{1 + ce^{2\alpha\phi}} \ , \tag{2.28}$$

which gives the Hubble parameter

$$H(t) = \frac{H_i + H_l c e^{2\alpha t}}{1 + ce^{2\alpha t}} \ . \tag{2.29}$$

Here H_i , H_l , c , and α are positive constants. In the early universe ($t \to -\infty$), we find that H becomes a constant $H \to H_i$ and at late times ($t \to +\infty$), H becomes a constant again $H \to H_l$. Then H_i could be regarded as the effective cosmological constant driving inflation, while H_l could be a small effective constant generating the late acceleration. Then, we should assume $H_i \gg H_l$. Hence, if we consider the model with action

$$\begin{aligned}
S &= \int d^4x\sqrt{-g}\left\{\frac{R}{2\kappa^2} - \frac{\omega(\phi)}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\right\} \ , \\
\omega(\phi) &\equiv -\frac{f'(\phi)}{\kappa^2} = \frac{2\alpha\left(H_i - H_l\right) c e^{2\alpha\phi}}{\kappa^2\left(1 + ce^{2\alpha\phi}\right)^2} \ , \\
V(\phi) &\equiv \frac{3f(\phi)^2 + f'(\phi)}{\kappa^2} = \frac{3H_i^2 + \left\{6H_iH_l - 2\alpha\left(H_i - H_l\right)\right\} c e^{2\alpha\phi} + c^2 H_l^2 e^{4\alpha\phi}}{\kappa^2\left(1 + ce^{2\alpha\phi}\right)^2} \ ,
\end{aligned} \tag{2.30}$$

we can realize the Hubble rate given by (2.29) with $\phi = t$. If we redefine the scalar field as

$$\varphi \equiv \int d\phi \sqrt{\omega(\phi)} = \frac{\mathrm{e}^{\alpha\phi}}{\kappa}\sqrt{\frac{2(a-b)c}{\alpha}} \ , \tag{2.31}$$

the action S in (2.30) can be rewritten in the canonical form

$$S = \int d^4x\sqrt{-g}\left\{\frac{R}{2\kappa^2} - \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \tilde{V}(\varphi)\right\} \ , \tag{2.32}$$

equation where

$$\tilde{V}(\varphi) = V(\phi) = \frac{3H_i^2 + \frac{\kappa^2\alpha\{6H_iH_l - 2\alpha(H_i-H_l)\}}{2(H_i-H_l)}\varphi^2 + \frac{3\kappa^4H_l^2\alpha^2}{4(H_i-H_l)^2}\varphi^4}{\kappa^2\left(1+\frac{\kappa^2\alpha}{2(H_i-H_l)}\varphi^2\right)^2} \ . \tag{2.33}$$

One should note that $\phi \to -\infty$ corresponds to $\varphi \to 0$ and $V \sim 3H_i^2/\kappa^2$, while $\phi \to +\infty$ corresponds to $\varphi \to \infty$ and $V \sim 3H_l^2/\kappa^2$, as expected. At early times ($\varphi \to 0$), $\tilde{V}(\varphi)$ behaves as

$$\tilde{V}(\varphi) \sim \frac{3H_i^2}{\kappa^2}\left\{1 - \frac{\kappa^2\alpha\left(3H_i+\alpha\right)}{3H_i^2}\varphi^2 + \mathcal{O}\left(\varphi^2\right)\right\} \ . \tag{2.34}$$

At early times, $\phi < 0$ and therefore, from Eq. (2.31), we find $\kappa\varphi\sqrt{\alpha/2\left(H_i-H_l\right)c} \ll 1$, from which it follows that

$$\begin{aligned}
&\frac{1}{3\kappa^2}\frac{\tilde{V}'(\varphi)}{\tilde{V}(\varphi)^2} \sim \frac{4\alpha^2\kappa^2\left(3H_i+\alpha\right)^2}{27H_i^4}\varphi^2 < \frac{8\alpha\left(3H_i+\alpha\right)^2\left(H_i-H_l\right)c}{27H_i^4} \ , \\
&\frac{1}{3\kappa^2}\frac{\left|\tilde{V}''(\varphi)\right|}{\tilde{V}(\varphi)} \sim \frac{2\left(3H_i+\alpha\right)\alpha}{9H_i^2} \ .
\end{aligned} \tag{2.35}$$

Then, if $\alpha \ll H_i$, the slow-roll conditions can be satisfied.

We may include matter with constant EoS parameter w_m . Then $\omega(\phi)$ and $V(\phi)$ are modified as

$$\begin{aligned}
\omega(\phi) &\equiv -\frac{f'(\phi)}{\kappa^2} - \frac{w_m+1}{2}g_0\mathrm{e}^{-3(1+w_m)g(\phi)} \ , \\
V(\phi) &\equiv \frac{3f(\phi)^2+f'(\phi)}{\kappa^2} + \frac{w_m-1}{2}g_0\mathrm{e}^{-3(1+w_m)g(\phi)} \ , \\
g(\phi) &\equiv \int d\phi f(\phi) = H_l\phi + \frac{H_i-H_l}{2}\ln\left(c+\mathrm{e}^{-2\alpha\phi}\right) \ .
\end{aligned} \tag{2.36}$$

The matter energy density is then given by

$$\rho_m = \rho_0 a^{-3(1+w_m)} = g_0\mathrm{e}^{-3(1+w_m)g(t)} = g_0\left(c+\mathrm{e}^{-2\alpha t}\right)^{-3(1+w_m)(H_i-H_l)/2}\mathrm{e}^{-3(1+w_m)H_lt} \ . \tag{2.37}$$

In the early universe $t \to -\infty$, ρ_m behaves as

$$\rho_m \sim g_0\mathrm{e}^{3(1+w_m)(2\alpha-H_l)t} \ . \tag{2.38}$$

On the other hand, the energy density of the scalar field behaves as

$$\rho_\varphi = \frac{1}{2}\dot{\varphi}^2 + \tilde{V}(\varphi) = \frac{\omega(\phi)}{2}\dot{\phi}^2 + V(\phi) \to \frac{3H_i^2}{\kappa^2} \ . \tag{2.39}$$

Then, if $2\alpha < H_l$ (and $w_m > -1$), the matter contribution could be neglected in comparison with the scalar field contribution.

Now let the present time be $t = t_0$. Then, we find that

$$\begin{aligned} \Omega_m &\equiv \frac{\kappa^2 \rho_m}{3H^2} \\ &= \frac{\kappa^2 g_0 \mathrm{e}^{4\alpha t_0} \left(c + \mathrm{e}^{-2\alpha t_0}\right)^{-3(1+w_m)(H_i - H_l)/2\alpha + 2} \mathrm{e}^{-3(1+w_m)H_l t_0}}{3\left(H_i + H_l c \mathrm{e}^{2\alpha t_0}\right)^2} , \end{aligned} \tag{2.40}$$

and $\Omega_\phi = 1 - \Omega_m$. If we assume $\alpha t_0 \gg 1$, we find

$$\Omega_m \sim \frac{\kappa^2 g_0}{3H_l^2} c^{-3(1+w_m)(H_i - H_l)/2\alpha} \mathrm{e}^{-3(1+w_m)H_l t_0} . \tag{2.41}$$

Hence, we may choose the parameters so that $\Omega_m \sim 0.27$, which could be consistent with the observed data. This model provides a quite realistic picture of the unification of the inflation with the present cosmic speed-up.

2.2 Accelerated expansion in the non-minimally curvature-coupled scalar theory

In the preceding section we have considered an action, (2.1), in which the scalar field is minimally coupled to gravity. In the present section, the scalar field couples to gravity through the Ricci scalar (see [142] for a review on cosmological applications). We begin from the action

$$S = \int d^4x \sqrt{-g} \left[(1 + f(\phi)) \frac{R}{\kappa^2} - \frac{1}{2} \omega(\phi) \partial_\mu \phi \partial^\mu \phi - V(\phi) \right] , \tag{2.42}$$

where $f(\phi)$ is an arbitrary function of the scalar field ϕ. Then, the effective gravitational coupling depends on ϕ, as $\kappa_{eff} = \kappa[1 + f(\phi)]^{-1/2}$. One can work in the Einstein frame, by performing the scale transformation

$$g_{\mu\nu} = [1 + f(\phi)]^{-1} \widetilde{g}_{\mu\nu} . \tag{2.43}$$

The tilde over g denotes an Einstein frame quantity. Thus, the action (2.42) in such a frame assumes the form [147]

$$S = \int d^4x \sqrt{-\widetilde{g}} \left\{ \frac{\widetilde{R}}{2\kappa^2} - \left[\frac{\omega(\phi)}{2(1 + f(\phi))} + \frac{6}{\kappa^2 (1 + f(\phi))} \left(\frac{d(1 + f(\phi)^{1/2})}{d\phi} \right)^2 \right] \partial_\mu \phi \partial^\mu \phi - \frac{V(\phi)}{[1 + f(\phi)]^2} \right\} . \tag{2.44}$$

The kinetic function can be written as $W(\phi) = \frac{\omega(\phi)}{1+f(\phi)} + \frac{3}{\kappa^2(1+f(\phi))^2} \left(\frac{df(\phi)}{d\phi}\right)^2$, and the extra term in the scalar potential can be absorbed by defining the new potential $U(\phi) = \frac{V(\phi)}{[1+f(\phi)]^2}$, so that we recover the action (2.1) in the Einstein frame, namely

$$S = \int dx^4 \sqrt{-\widetilde{g}} \left(\frac{\widetilde{R}}{\kappa^2} - \frac{1}{2} W(\phi) \, \partial_\mu \phi \partial^\mu \phi - U(\phi) \right) . \tag{2.45}$$

We assume that the metric is FLRW and spatially flat in this frame

$$d\widetilde{s}^2 = -d\widetilde{t}^2 + \widetilde{a}^2(\widetilde{t}) \sum_i dx_i^2 \ , \tag{2.46}$$

then, the equations of motion in this frame are given by

$$\widetilde{H}^2 = \frac{\kappa^2}{6}\rho_\phi \ , \tag{2.47}$$

$$\dot{\widetilde{H}} = -\frac{\kappa^2}{4}\left(\rho_\phi + p_\phi\right) \ , \tag{2.48}$$

$$\frac{d^2\phi}{d\tilde{t}^2} + 3\tilde{H}\frac{d\phi}{d\tilde{t}} + \frac{1}{2W(\phi)}\left[W'(\phi)\left(\frac{d\phi}{d\tilde{t}}\right)^2 + 2U'(\phi)\right] = 0 \ , \tag{2.49}$$

where $\rho_\phi = \frac{1}{2}W(\phi)\dot{\phi}^2 + U(\phi)$, $p_\phi = \frac{1}{2}W(\phi)\dot{\phi}^2 - U(\phi)$, and the Hubble parameter is $\widetilde{H} \equiv \frac{1}{\tilde{a}}\frac{d\tilde{a}}{d\tilde{t}}$. Then,

$$W(\phi)\dot{\phi}^2 = -4\dot{\widetilde{H}} \ , \qquad U(\phi) = 6\widetilde{H}^2 + 2\dot{\widetilde{H}}. \tag{2.50}$$

Note that $\dot{\widetilde{H}} > 0$ is equivalent to $W < 0$; superacceleration is due to the "wrong" (negative) sign of the kinetic energy, which is the distinctive feature of a phantom field. The scalar field could be redefined to eliminate the factor $W(\phi)$, but this would not correct the sign of the kinetic energy.

If we choose $W(\phi)$ and $U(\phi)$ as $\omega(\phi)$ and $V(\phi)$ in (2.7),

$$W(\phi) = -\frac{2}{\kappa^2}g'(\phi) \ , \quad U(\phi) = \frac{1}{\kappa^2}\left[3g(\phi)^2 + g'(\phi)\right] \ , \tag{2.51}$$

by using a function $g(\phi)$ instead of $f(\phi)$ in (2.7), we find a solution as in (2.8),

$$\phi = \widetilde{t} \ , \quad \widetilde{H}(\widetilde{t}) = g(\widetilde{t}) \ . \tag{2.52}$$

In (2.51) and hereafter in this section, we have dropped the matter contribution for simplicity.

We consider the de Sitter solution in this frame,

$$\widetilde{H} = \widetilde{H}_0 = \text{const.} \ \rightarrow \ \widetilde{a}(\widetilde{t}) = \widetilde{a}_0 e^{\widetilde{H}_0\widetilde{t}} \ . \tag{2.53}$$

We will see below that accelerated expansion can be obtained in the original frame corresponding to the Einstein frame (2.46) with the solution (2.53), by choosing an appropriate function $f(\phi)$. From (2.53) and the definition of $W(\phi)$ and $U(\phi)$, we have

$$W(\phi) = 0 \ \rightarrow \ \omega(\phi) = -\frac{3}{[1+f(\phi)]\,\kappa^2}\left[\frac{df(\phi)}{d\phi}\right]^2 \ , \quad U(\phi) = \frac{6}{\kappa^2}\widetilde{H}_0^2 \ \rightarrow \ V(\phi) = \frac{6}{\kappa^2}\widetilde{H}_0^2[1+f(\phi)]^2 \ . \tag{2.54}$$

Thus, the scalar field has a non-canonical kinetic term in the original frame, while in the Einstein frame the latter can be positive, depending on $W(\phi)$. The correspondence between conformal frames can be made explicit through the conformal transformation (2.43). Assuming a spatially flat FLRW metric in the original frame,

$$ds^2 = -dt^2 + a^2(t)\sum_{i=1}^{3} dx_i^2 \ , \tag{2.55}$$

then, the relation between the time coordinate and the scale parameter in these frames is given by

$$t = \int \frac{d\widetilde{t}}{([1+f(\widetilde{t})]^{1/2}} \ , \quad a(t) = [1+f(\widetilde{t})]^{-1/2}\,\widetilde{a}(\widetilde{t}) \ . \tag{2.56}$$

Now let us discuss the late-time acceleration in the model under discussion.

Example 1

As a first example, we consider the coupling function between the scalar field and the Ricci scalar

$$f(\phi) = \frac{1-\alpha\phi}{\alpha\phi} \ , \tag{2.57}$$

where α is a constant. Then, from (2.54), the kinetic function $\omega(\phi)$ and the potential $V(\phi)$ are

$$\omega(\phi) = -\frac{3}{\kappa^2\alpha^2}\frac{1}{\phi^3} \ , \qquad V(\phi) = \frac{6\widetilde{H}_0}{\kappa^2\alpha^2}\frac{1}{\phi^2} \ , \tag{2.58}$$

respectively. The solution for the current example is found to be

$$\phi(t) = \widetilde{t} = \frac{1}{\alpha}\left(\frac{3\alpha}{2}\,t\right)^{2/3} \ , \qquad a(t) = \widetilde{a}_0\left(\frac{3\alpha}{2}\,t\right)^{1/3}\exp\left[\frac{\widetilde{H}_0}{\alpha}\left(\frac{3\alpha}{2}\,t\right)^{2/3}\right] \ . \tag{2.59}$$

We now calculate the acceleration parameter to study the behavior of the scalar parameter in the original frame,

$$\frac{\ddot{a}}{a} = -\frac{2}{9}\frac{1}{t^2} + \widetilde{H}_0\left(\frac{2}{3\alpha}\right)^{1/3}\left[\frac{1}{t^{4/3}} + \widetilde{H}_0\left(\frac{2}{3\alpha}\right)^{1/3}\frac{1}{t^{2/3}}\right] \ . \tag{2.60}$$

We observe that for small values of t the acceleration is negative; after that we get accelerated expansion for large t; finally, the universe ends with zero acceleration as $t \to \infty$. Thus, late time accelerated expansion is reproduced by the action (2.42) with the function $f(\phi)$ given by Eq. (2.57).

Example 2

As a second example, consider the function

$$f(\phi) = \phi - t_0 \ . \tag{2.61}$$

From (2.56), the kinetic term and the scalar potential are, in this case,

$$\omega(\phi) = -\frac{3}{\kappa^2}\frac{1}{(1+\phi-t_0)} \ , \qquad V(\phi) = \frac{6\widetilde{H}_0^2}{\kappa^2}(1+\phi-t_0) \ . \tag{2.62}$$

The solution in the original (Jordan) frame reads

$$\phi(t) = \frac{t^2}{4} + t_0 - 1 \ , \quad a(t) = \frac{2\widetilde{a}_0}{t}\exp\left[\widetilde{H}_0\left(\frac{t^2}{4} + t_0 - 1\right)\right] \ , \tag{2.63}$$

and the corresponding acceleration is

$$\frac{\ddot{a}}{a} = \frac{1}{t^2}\left[\frac{\widetilde{H}_0 t^2}{2} + \left(\frac{\widetilde{H}_0 t^2}{2} - 1\right)^2 + 1\right] \ . \tag{2.64}$$

Notice that this solution describes acceleration at every time t and, for $t \to \infty$, the acceleration tends to a constant value, as in de Sitter spacetime, hence similar to what happens in the Einstein frame. Thus, we have proved here that it is possible to reproduce accelerated expansion in both frames, by choosing a convenient function for the coupling $f(\phi)$.

2.2.1 Chameleon mechanism

The kind of theories described by the above action (2.42) have problems when matter is included and one perform a conformal transformation and the Einstein frame is recovered, then the local gravity tests may be violated by a faith force that appeared on a test particle and the violation of the Equivalence Principle is presented. This kind of problems are well constrained by the experiments to a certain value of the coupling parameter as it is pointed below. Recently, a very interesting idea originally proposed in Ref. [183] avoids the constrains from local gravity tests in such a way that the effects of the scalar field are neglible at small scales but it acquires an important role for large scales, whose effects may produce the current acceleration of the Universe
Let us start by rewriting the action in the Einstein frame (2.44) in a similar form as the original Brans-Dicke action by redefining the scalar field ϕ and rewriting the kinetic term $\omega(\phi)$ in terms of the coupling $(1+f(\phi))$, then the action is given by:

$$S_E = \int d^4x\sqrt{-\tilde{g}}\left[\frac{\tilde{R}}{2\kappa^2} - \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma - U(\sigma) + \mathrm{e}^{4\beta\sigma}\widetilde{L_m}\right] , \tag{2.65}$$

where:

$$\mathrm{e}^{-2\beta\sigma} = 1 + f(\phi) , \tag{2.66}$$

here β is a constant. As it is seen in the action (2.65), the matter density lagrangian couples to the scalar field σ, such that a massive test particle will be under a fifth force, and the equation of motion will be:

$$\ddot{x}^\mu + \tilde{\Gamma}^\mu_{\lambda\nu}\dot{x}^\lambda\dot{x}^\nu = -\beta\partial^\mu\sigma , \tag{2.67}$$

where x^μ represents the four-vector describing the path of a test particle moving in the metric $\tilde{g}^(\mu\nu)$, and $\tilde{\Gamma}^\mu_{\nu\lambda}$ are the Christoffel symbols for the metric $\tilde{g}^{\mu\nu}$. From the equation (2.67), the scalar field σ can be seen as a potential from a force given by:

$$F_\sigma = -M\beta\partial^\mu\sigma , \tag{2.68}$$

where M is the mass of a test particle. Then, this kind of theories reproduces a fifth force which it has been tested by the experiments to a limit $\beta < 1.6\times 10^{-3}$ (Ref. [30]). The aim of the chameleon mechanism is that it makes the fifth force neglible for small scales passing the local tests. Such mechanism works in the following way, by varying the action (2.65) with respect the scalar field σ, the equation of motion for the scalar field is obtained:

$$\bigtriangledown^2\sigma = U_{,\sigma} - \beta\mathrm{e}^{4\beta\sigma}\tilde{g}^{\mu\nu}\tilde{T}_{\mu\nu} , \tag{2.69}$$

where the energy-momentum tensor is given by $\tilde{T}^{\mu\nu} = \frac{2}{\sqrt{-\tilde{g}}}\frac{\partial L_m}{\partial\tilde{g}^{\mu\nu}}$. For simplicity we restrict to dust matter $\tilde{g}^{\mu\nu}\tilde{T}_{\mu\nu} = -\tilde{\rho}_m$, where the energy density may be written in terms of the conformal transformation (2.66) as $\rho_m = \tilde{\rho}_m\mathrm{e}^{3\beta\sigma}$. Then, The equation for the scalar field (2.69) is written in the following way:

$$\bigtriangledown^2\sigma = U_{,\sigma} + \beta\rho_m\mathrm{e}^{\beta\sigma} , \tag{2.70}$$

here it is shown that the dynamics of the scalar field depends on the matter energy density. We may write the right side of the equation (2.70) as an effective potential $U_{eff} = U(\sigma) + \rho_m\mathrm{e}^{\beta\sigma}$. Then, the behavior of the scalar field will depend on the effective potential, and the solutions for the equation (2.70) are given by studying U_{eff}. By imposing to the scalar potential $U(\sigma)$ to be a monotonic decreasing function, the effective potential will have a minimum that will govern the solution for the scalar field (for more details see [183]), this minimum is given by:

$$U_{,\sigma}(\sigma_{min}) + \beta\rho_m\mathrm{e}^{\beta\sigma_{min}} = 0 , \tag{2.71}$$

which depends on the local matter density. At this minimum, the scalar field mass will be given by:

$$m_\sigma^2 = U_{,\sigma\sigma}(\sigma_{min}) + \beta^2 \rho_m \mathrm{e}^{\beta\sigma_{min}} \ . \tag{2.72}$$

Then, because of the characteristics of the scalar potential $U(\sigma)$, larger values of the local density ρ_m corresponds to small values of σ_{min} and large values of m_σ, so it is possible for sufficiently large values of the scalar field mass to avoid Equivalence Principle violations and fifth forces on the Earth. As the energy density ρ_m becomes smaller, the scalar field mass m_σ decreases and σ_{min} increases, such that at large scales (when $\rho \sim H_0^2$), the effects of the scalar field become detected, where the accelerated expansion of the Universe may be a possible effect. Then, one may restrict the original scalar potential $V(\phi)$ and the coupling $(1+f(\phi))$ trough the mechanism shown above. The effective potential U_{eff} is written in terms of ϕ by the equation (2.66) as:

$$U_{eff}(\phi) = U(\phi) + \frac{\rho_m}{(1+f(\phi))^{1/2}} \ , \tag{2.73}$$

where $U(\phi) = V(\phi)/(1+f(\phi))$. Then, by giving a function $f(\phi)$ and a scalar potential $V(\phi)$, one may construct using the conditions described above on the mass m_σ, a cosmological model that reproduces the current accelerated expansion and at the same time, avoid the local test of gravity.

2.3 Reconstruction of non-minimally coupled scalar field theory

We now consider the reconstruction problem in the non-minimally coupled scalar field theory, or the Brans-Dicke theory. We begin with the same scalar-tensor theory with constant parameters ϕ_0 and V_0:

$$S = \int d^4x\sqrt{-g}\left[\frac{R}{2\kappa^2} - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\right] \ , \qquad V(\phi) = V_0\,\mathrm{e}^{-2\phi/\phi_0} \ , \tag{2.74}$$

which admits the exact solution

$$\phi = \phi_0 \ln\left|\frac{t}{t_1}\right| \ , \quad H = \frac{\kappa^2\phi_0^2}{2t} \ , \quad t_1^2 \equiv \frac{\gamma\phi_0^2\left(\frac{3\gamma\kappa^2\phi_0^2}{2} - 1\right)}{2V_0} \ . \tag{2.75}$$

We choose $\phi_0^2\kappa^2 > 2/3$ and $V_0 > 0$ so that $t_1^2 > 0$. For this solution, the metric is given by

$$ds^2 = -dt^2 + a_0^2\left(\frac{t}{t_0}\right)^{\kappa^2\phi_0^2} \sum_{i=1}^{3}\left(dx^i\right)^2 \ , \tag{2.76}$$

which can be transformed into the conformal form:

$$ds^2 = a_0^2\left(\frac{\tau}{\tau_0}\right)^{-\frac{\kappa^2\phi_0^2}{\frac{\kappa^2\phi_0^2}{2}-1}} \left(-d\tau^2 + \sum_{i=1}^{3}\left(dx^i\right)^2\right) \ . \tag{2.77}$$

Here

$$\frac{\tau}{\tau_0} = -\left(\frac{t}{t_0}\right)^{-\frac{\kappa^2\phi_0^2}{2}+1} \ , \quad \tau_0 \equiv \frac{t_0}{a_0\left(\frac{\kappa^2\phi_0^2}{2} - 1\right)} \ . \tag{2.78}$$

and therefore, by using (2.75), one finds

$$\frac{\tau}{\tau_0} = -\left(\frac{t_1}{t_0}\right)^{-\frac{\kappa^2}{2}+1} \mathrm{e}^{-\frac{1}{-\frac{\kappa^2}{2}+1}\frac{\phi}{\phi_0}} \ . \tag{2.79}$$

We now consider an arbitrary cosmology given by the metric

$$d\tilde{s}^2 = f(\tau) \left(-d\tau^2 + \sum_{i=1}^{3} \left(dx^i\right)^2 \right) , \tag{2.80}$$

where τ is the conformal time. Since

$$ds^2 = \frac{1}{f(\tau)} a_0^2 \left(\frac{\tau}{\tau_0}\right)^{-\frac{\kappa^2\phi_0^2}{\frac{\kappa^2\phi_0^2}{2}-1}} d\tilde{s}^2 = \mathrm{e}^{\varphi} d\tilde{s}^2 , \quad \mathrm{e}^{\varphi} \equiv \frac{a_0^2 \left(\frac{t_1}{t_0}\right)^{\kappa^2\phi_0^2} \mathrm{e}^{\kappa^2\phi_0\phi}}{f\left(-\tau_0 \left(\frac{t_1}{t_0}\right)^{-\frac{\kappa^2}{2}+1} \mathrm{e}^{-\frac{1}{-\frac{\kappa^2}{2}+1}\frac{\phi}{\phi_0}} \right)} , \tag{2.81}$$

if we begin with the action in which $g_{\mu\nu}$ in (2.74) is replaced by $\mathrm{e}^{\varphi}\tilde{g}_{\mu\nu}$,

$$S = \int d^4x \sqrt{-\tilde{g}}\, \mathrm{e}^{\varphi(\phi)} \left\{ \frac{R}{2\kappa^2} - \frac{1}{2}\left[1 - 3\left(\frac{d\varphi}{d\phi}\right)^2\right] \partial_\mu\phi\partial^\mu\phi - \mathrm{e}^{\varphi(\phi)} V(\phi) \right\} , \tag{2.82}$$

we obtain the solution (2.80).

Example

By using the conformal time τ , the metric of de Sitter space

$$ds^2 = -dt^2 + \mathrm{e}^{2H_0 t} \sum_{i=1}^{3} \left(dx^i\right)^2 , \tag{2.83}$$

can be rewritten as

$$ds^2 = \frac{1}{H_0^2\tau^2} \left(-d\tau^2 + \sum_{i=1}^{3} \left(dx^i\right)^2 \right) . \tag{2.84}$$

Here τ is related to t by $\mathrm{e}^{-H_0 t} = -H_0\tau$. Then $t \to -\infty$ corresponds to $\tau \to +\infty$ and $t \to +\infty$ corresponds to $\tau \to 0$.

As an example of $f(\tau)$ in (2.80), we may consider

$$f(\tau) = \frac{\left(1 + H_L^2\tau^2\right)}{H_L^2\tau^2\left(1 + H_I^2\tau^2\right)} , \tag{2.85}$$

where H_L and H_I are constants. At early times in the history of the universe $\tau \to \infty$ (corresponding to $t \to -\infty$), $f(\tau)$ behaves as

$$f(\tau) \to \frac{1}{H_I^2\tau^2} . \tag{2.86}$$

Then the Hubble rate is given by a constant H_I , and therefore the universe is asymptotically de Sitter space, corresponding to inflation. On the other hand, at late times $\tau \to 0$ (corresponding to $t \to +\infty$), $f(\tau)$ behaves as

$$f(\tau) \to \frac{1}{H_L^2\tau^2} . \tag{2.87}$$

Then the Hubble rate is again a constant H_L , which may correspond to the late time acceleration of the universe. This proves that our reconstruction program can be applied directly to the non-minimally coupled scalar theory.

2.4 Late time acceleration and inflation with several scalar fields

In this section we begin by considering a model with two scalar fields minimally coupled to gravity (see [133, 231, 232]). Such models are used, for example, in reheating scenarios after inflation.

An additional degree of freedom appears in this case, so that for a given solution we may choose different conditions on the scalar fields, as shown below. It is possible to restrict these conditions by studying the perturbative regime for each solution. The action we consider is

$$S = \int \sqrt{-g}\left[\frac{R}{2\kappa^2} - \omega(\phi)\partial_\mu\phi\partial^\mu\phi - \sigma(\chi)\partial_\mu\chi\partial^\mu\chi - V(\phi,\chi)\right] , \tag{2.88}$$

where $\omega(\phi)$ and $\sigma(\chi)$ are the kinetic terms, which depend on the fields ϕ and χ, respectively. We again assume a flat FLRW metric. The Friedmann equations are written as

$$H^2 = \frac{\kappa^2}{3}\left[\frac{1}{2}\omega(\phi)\dot{\phi}^2 + \frac{1}{2}\sigma(\chi)\dot{\chi}^2 + V(\phi,\chi)\right] , \quad \dot{H} = -\frac{\kappa^2}{2}\left[\omega(\phi)\dot{\phi}^2 + \sigma(\chi)\dot{\chi}^2\right] . \tag{2.89}$$

By means of a convenient transformation, we can always redefine the scalar fields so that we can write $\phi = \chi = t$. The scalar field equations are given by

$$\omega(\phi)\ddot{\phi} + \frac{1}{2}\omega'(\phi)\dot{\phi}^2 + 3H\omega(\phi)\dot{\phi} + \frac{\partial V(\phi,\chi)}{\partial\phi} = 0 , \quad \sigma(\chi)\ddot{\chi} + \frac{1}{2}\sigma'(\chi)\dot{\chi}^2 + 3H\sigma(\chi)\dot{\chi} + \frac{\partial V(\phi,\chi)}{\partial\chi} = 0 . \tag{2.90}$$

Then, for a given solution $H(t) = f(t)$, and combining the first Friedmann equation with each scalar field equation, respectively, we find

$$\omega(\phi) = -\frac{2}{\kappa^2}\frac{\partial f(\phi,\chi)}{\partial\phi} , \quad \sigma(\chi) = -\frac{2}{\kappa^2}\frac{\partial f(\phi,\chi)}{\partial\chi} , \tag{2.91}$$

where the function $f(\phi,\chi)$ carries down to $f(t,t) \equiv f(t)$, and is defined as

$$f(\phi,\chi) = -\frac{\kappa^2}{2}\left[\int \omega(\phi)d\phi + \int \sigma(\chi)d\chi\right] . \tag{2.92}$$

The scalar potential can be expressed as

$$V(\phi,\chi) = \frac{1}{\kappa^2}\left[3f(\phi,\chi)^2 + \frac{\partial f(\phi,\chi)}{\partial\phi} + \frac{\partial f(\phi,\chi)}{\partial\chi}\right] , \tag{2.93}$$

and the second Friedmann equation reads

$$-\frac{2}{\kappa^2}f'(t) = \omega(t) + \sigma(t). \tag{2.94}$$

Then, the kinetic functions may be chosen to be

$$\omega(\phi) = -\frac{2}{\kappa^2}\left[f'(\phi) + g(\phi)\right] , \quad \sigma(\phi) = \frac{2}{\kappa^2}g(\chi) , \tag{2.95}$$

where g is an arbitrary function. Hence, the scalar field potential is finally obtained as

$$V(\phi,\chi) = \frac{1}{\kappa^2}\left[3f(\phi,\chi)^2 + f'(\phi) + g(\phi) - g(\chi)\right] . \tag{2.96}$$

Example 1

We can consider again the solution (2.17)

$$f(t) = \frac{H_0}{t_s - t} + \frac{H_1}{t^2} \ . \tag{2.97}$$

This solution, as already seen in Sec. II, reproduces unified inflation and late time acceleration in a scalar field model with matter given by action (2.1). We may now understand this solution as derived from the two-scalar field model (2.88), where a degree of freedom is added so that we can choose various types of scalar kinetic and potential terms, as shown below for the solution (2.97). Then, from Eqs. (2.95) and (2.97), the kinetic terms follow:

$$\omega(\phi) = -\frac{2}{\kappa^2}\left[\frac{H_0}{(t_s-\phi)^2} - \frac{2H_1}{\phi^3} + g(\phi)\right] \ , \quad \sigma(\phi) = \frac{2}{\kappa^2} g(\chi) \ . \tag{2.98}$$

The function $f(\phi,\chi)$ is

$$f(\phi,\chi) = \frac{H_0}{t_s-\phi} + \frac{H_1}{\phi^2} + \int d\phi\, g(\phi) - \int d\chi\, g(\chi), \tag{2.99}$$

while the scalar potential is

$$V(\phi,\chi) = \frac{1}{\kappa^2}\left[3f(\phi,\chi)^2 + \frac{H_0}{(t_s-\phi)^2} - \frac{2H_1}{\phi^3} + g(\phi) - g(\chi)\right] \ . \tag{2.100}$$

This scalar potential reproduces the solution (2.97) with an arbitrary function $g(t)$.

It is possible to further restrict $g(t)$ by studying the stability of the system considered. To this end, we define the functions

$$X_\phi = \dot{\phi} \ , \quad X_\chi = \dot{\chi} \ , \quad Y = \frac{f(\phi,\chi)}{H} \ . \tag{2.101}$$

Then, the Friedmann and scalar field equations can be written as

$$\frac{dX_\phi}{dN} = -\frac{1}{2H}\frac{\omega'(\phi)}{\omega(\phi)}\left(X_\phi^2 - 1\right) - 3\left(X_\phi - Y\right) \ , \tag{2.102}$$

$$\frac{dX_\sigma}{dN} = -\frac{1}{2H}\frac{\sigma'(\chi)}{\sigma(\chi)}\left(X_\chi^2 - 1\right) - 3\left(X_\chi - Y\right) \ ,$$

$$\frac{dY}{dN} = \frac{\kappa^2}{2H^2}\left[\omega(\phi)X_\phi(YX_\phi - 1) + \sigma(\chi)X_\chi(YX_\chi - 1)\right] \ , \tag{2.103}$$

where $\frac{d}{dN} = \frac{1}{H}\frac{1}{dt}$. At $X_\phi = X_\chi = Y = 1$, we consider the perturbations

$$X_\phi = 1 + \delta X_\phi \ , \quad X_\chi = 1 + \delta X_\chi \ , \quad Y = 1 + \delta Y, \tag{2.104}$$

then

$$\frac{d}{dN}\begin{pmatrix}\delta X_\phi \\ \delta X_\chi \\ \delta Y\end{pmatrix} = M \begin{pmatrix}\delta X_\phi \\ \delta X_\chi \\ \delta Y\end{pmatrix} \ , \qquad M = \begin{pmatrix} -\frac{\omega'(\phi)}{H\omega(\phi)} - 3 & 0 & 3 \\ 0 & -\frac{\sigma'(\chi)}{H\sigma(\chi)} - 3 & 3 \\ \kappa^2\frac{\omega(\phi)}{2H^2} & \kappa^2\frac{\sigma(\chi)}{2H^2} & \kappa^2\frac{\omega(\phi)+\sigma(\chi)}{2H^2} \end{pmatrix} . \tag{2.105}$$

The eigenvalue equation is given by

$$\left(\frac{\omega'(\phi)}{H\omega(\phi)}+3+\lambda\right)\left(\frac{\sigma'(\chi)}{H\sigma(\chi)}+3+\lambda\right)\left(\frac{\kappa^2}{2H^2}(\omega(\phi)+\sigma(\phi))-\lambda\right)$$
$$+\frac{3\kappa^2\omega(\phi)}{2H^2}\left(\frac{\sigma'(\chi)}{H\sigma(\chi)}+3+\lambda\right)+\frac{3\kappa^2\sigma(\chi)}{2H^2}\left(\frac{\omega'(\phi)}{H\omega(\phi)}+3+\lambda\right)=0\ . \tag{2.106}$$

To avoid divergences in the eigenvalues, we choose the kinetic functions to satisfy

$$\omega(\phi)\neq 0\ ,\quad \sigma(\chi)\neq 0\ , \tag{2.107}$$

hence, the eigenvalues in Eq. (2.106) are finite. Summing up, under these conditions, the solution (2.97) has no infinite instability when the transition from the non-phantom to the phantom phase occurs.

As an example, we may choose $g(t)=\alpha/t^3$, where α is a constant that satisfies $\alpha>2H_1$. Then, the $f(\phi,\chi)$ function (2.99) is given by

$$f(\phi,\chi)=\frac{H_0}{t_s-\phi}-\frac{(\alpha-2H_1)}{2\phi^2}+\frac{\alpha}{2\chi^2}\ . \tag{2.108}$$

As a result, the kinetic terms (2.98) are expressed as

$$\omega(\phi)=-\frac{2}{\kappa^2}\left[\frac{H_0}{(t_s-\phi)^2}-\frac{2H_1}{\phi^3}+\frac{\alpha}{\phi^3}\right]\ ,\quad \sigma(\chi)=\frac{2}{\kappa^2}\frac{\alpha}{\chi^3}\ , \tag{2.109}$$

and the potential reads

$$V(\phi,\chi)=\frac{1}{\kappa^2}\left[3f(\phi,\chi)^2+\frac{H_0}{(t_s-\phi)^2}+\frac{\alpha-2H_1}{\phi^3}-\frac{\alpha}{\chi^3}\right]\ . \tag{2.110}$$

This potential reproduces the solution (2.97) that unifies inflation and late time acceleration in the context of scalar-tensor theories, involving two scalar fields. Notice that the extra degree of freedom gives the possibility to select a different kinetic and scalar potential in such a manner that we get the same solution.

In the case in which the condition (2.107) is not imposed, the kinetic terms (2.95) may have zeros for $0<t<t_s$, so that the perturbation analysis performed above ceases to be valid because some of the eigenvalues could diverge.

2.4.1 General case: n scalar fields

As a generalization of the action (2.88), we now consider the corresponding one for n scalar fields,

$$S=\int d^4x\sqrt{-g}\left[\frac{1}{2\kappa^2}R-\frac{1}{2}\sum_{i=1}^{n}\omega_i(\phi_i)\partial_\mu\phi_i\partial^\mu\phi_i-V(\phi_1,\phi_2,\cdots,\phi_n)\right]\ . \tag{2.111}$$

The associated Friedmann equations are

$$H^2=\frac{\kappa^2}{3}\left[\sum_{i=1}^{n}\frac{1}{2}\omega_i(\phi_i)\dot{\phi}_i^{\ 2}+V(\phi_1,\cdots,\phi_n)\right]\ ,\quad \dot{H}=-\frac{\kappa^2}{2}\left[\sum_{i=1}^{n}\omega_i(\phi_i)\dot{\phi}_i^{\ 2}\right]\ . \tag{2.112}$$

We can proceed analogously to the case of two scalar fields so that the kinetic terms are written as

$$\sum_{i=1}^{n} \omega_i(t) = -\frac{2}{\kappa^2} f'(t) \ , \tag{2.113}$$

hence,

$$\omega_i(\phi_i) = -\frac{2}{\kappa^2}\frac{df(\phi_1, \cdots, \phi_n)}{d\phi_i} \ , \quad V(\phi_1, \cdots, \phi_n) = \frac{1}{\kappa^2}\left[3f(\phi_1, \cdots, \phi_n)^2 + \sum_{i=1}^{n}\frac{df(\phi_1, \cdots, \phi_n)}{d\phi_i}\right] \ , \tag{2.114}$$

where $f(t, t, \cdots, t) \equiv f(t)$. Then the following solution is found

$$\phi_i = t \ , \quad H(t) = f(t) \ . \tag{2.115}$$

From (2.113) we can choose, as done above, the kinetic terms to be

$$\omega_1(\phi_1) = -\frac{2}{\kappa^2}\left[f'(\phi_1) + g_2(\phi_1) + \cdots + g_n(\phi_1)\right] \ , \quad \omega_2(\phi_2) = \frac{2}{\kappa^2} g_2(\phi_2) \ , \cdots , \ \omega_n(\phi_n) = \frac{2}{\kappa^2} g_n(\phi_n) \ . \tag{2.116}$$

Then, there are $n-1$ arbitrary functions that reproduce the solution (2.115) so reconstruction may be successfully done. They could be chosen so that dark matter is also represented by some of the scalar fields appearing in the action (2.111).

2.5 Reconstruction of inflation and cosmic acceleration from two-scalar theory

In the present section, inflation and cosmic acceleration are reconstructed separately, by means of a two scalar field model that reproduces some of the cosmological constraints at each epoch. We explore an inflationary model in which the scalar potential, given for a pair of scalar fields, exhibits an extra degree of freedom and can be chosen in such way that slow-roll conditions are satisfied. Also, the cosmic acceleration is reproduced with a pair of scalar fields plus an ordinary matter term, in which the values of the observed cosmological density parameter ($\Omega_{\mathrm{DE}} \simeq 0.7$) and of the EoS parameter ($w_{\mathrm{DE}} \simeq -1$) are reproduced in a quite natural way. For the case of a single scalar, the reconstruction for similarly distant epochs was given in Ref. [216].

Inflation

In the previous sections, models describing inflation and late-time accelerated expansion have been constructed by using certain convenient scalar-tensor theories. In this section, we present an inflationary model with two scalar fields, which can be constructed in such a way that the inflationary conditions are carefully accounted for. For this purpose, we use some of the techniques given in the previous section. The action during the inflationary epoch is written as

$$S = \int \sqrt{-g}\left[\frac{R}{2\kappa^2} - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\partial_\mu\chi\partial^\mu\chi - V(\phi, \chi)\right] \ . \tag{2.117}$$

We will show that a general solution can be constructed, where the scalar field potential is not completely specified because of the extra degree of freedom represented by the second scalar field added, in a way

similar to the situation occurring in the previous section. Considering a spatially flat FLRW metric, the Friedmann and scalar field equations are obtained by using the Einstein equations and varying the action with respect to both scalar fields:

$$H^2 = \frac{\kappa^2}{3}\left[\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\dot{\chi}^2 + V(\phi,\chi)\right] , \quad \ddot{\phi} + 3H\dot{\phi} + \frac{\partial V(\phi,\chi)}{\partial \phi} = 0 , \quad \ddot{\chi} + 3H\dot{\chi} + \frac{\partial V(\phi,\chi)}{\partial \chi} = 0 . \quad (2.118)$$

We assume that the slow-roll conditions are satisfied, that is, $\ddot{\phi} \ll 3H\dot{\phi}$ $(\ddot{\chi} \ll 3H\dot{\chi})$, and $\dot{\phi}^2 \ll V(\phi,\chi)$ $\left(\dot{\chi}^2 \ll V(\phi,\chi)\right)$, in order for inflation to occur. Then, Eqs. (2.118) take the form

$$H^2 \approx \frac{\kappa^2}{3}V(\phi,\chi) , \quad 3H\dot{\phi} + \frac{\partial V(\phi,\chi)}{\partial \phi} \approx 0 , \quad 3H\dot{\chi} + \frac{\partial V(\phi,\chi)}{\partial \chi} \approx 0 \quad (2.119)$$

and the slow-roll conditions read

$$\frac{1}{3\kappa^2}\frac{V_{,i}V^{,i}}{V^2} \ll 1 , \quad \frac{1}{3\kappa^2}\frac{\sqrt{V_{,ij}V^{,ij}}}{V} \ll 1 . \quad (2.120)$$

Here, $V_{,i}$ denotes the partial derivative of V with respect to one of the scalar fields ($i = \phi, \chi$). As done previously, a scalar potential $V(\phi,\chi)$ can be constructed, although in this case the conditions for inflation need to be taken into account. From (2.119), the potential is given by

$$V(\phi,\chi) = \frac{3}{\kappa^2}\, H^2(\phi,\chi) . \quad (2.121)$$

We can choose this potential such that

$$V(\phi,\chi) = \frac{3}{\kappa^2}\left[f^2(\phi,\chi) + g_1(\phi) - g_2(\chi)\right] . \quad (2.122)$$

The three components f , g_1 , and g_2 are arbitrary functions, and $g(N) = g_1(N) = g_2(N)$ where, for convenience, we use the number of e-folds $N \equiv \ln \frac{a(t)}{a_i}$ instead of the cosmic time, and a_i denotes the initial value of the scale factor before inflation. Then, the following solution is found:

$$H(N) = f(N) . \quad (2.123)$$

Hence, Eqs. (2.119) may be expressed as a set of differential equations with the number of e-folds as independent variable,

$$3f^2(N)\frac{d\phi}{dN} + \frac{\partial V(\phi,\chi)}{\partial \phi} \approx 0 , \quad 3f^2(N)\frac{d\chi}{dN} + \frac{\partial V(\phi,\chi)}{\partial \chi} \approx 0 . \quad (2.124)$$

To illustrate this construction, let us use a simple example. The following scalar potential, as a function of the number of e-folds N , is considered:

$$V(\phi,\chi) = \frac{3}{\kappa^2}\left[H_0^2 N^{2\alpha}\right] , \quad (2.125)$$

where α and H_0 are free parameters. By specifying the arbitrary function $g(N)$, one can find a solution for the scalar fields. Let us choose, for the sake of simplicity, $g(N) = g_0 N^{2\alpha}$, where g_0 is a constant, and $f(\phi,\chi) = f(\phi)$ (*i.e.*, as a function of the scalar field ϕ only). Then, using Eqs. (2.124), the solutions for the scalar fields are found to be

$$\phi(N) = \phi_0 - \frac{1}{\kappa H_0}\sqrt{2\alpha(H_0^2 + g_0)}\, \ln N , \quad \chi(N) = \chi_0 - \frac{1}{\kappa H_0}\sqrt{2g_0\alpha N} , \quad (2.126)$$

and the scalar potential can be written as

$$V(\phi,\chi)=\frac{3}{\kappa^2}\left[(H_0+g_0)\exp\left(\kappa\frac{\sqrt{2\alpha}H_0}{\sqrt{H_0^2+g_0}}(\phi_0-\phi)\right)-g_0\left(\frac{\kappa^2H_0^2(\chi_0-\chi)^2}{2g_0\alpha}\right)^{2\alpha}\right]\ . \tag{2.127}$$

We are now able to impose the slow-roll conditions by evaluating the slow-roll parameters

$$\begin{aligned}\frac{1}{3\kappa^2}\frac{V_{,i}V^{,i}}{V^2} &= \frac{2\alpha}{3H_0^2}\left(\frac{(H_0+g_0)^2}{H_0^2+g_0}+\frac{4g_0}{N}\right)\ll 1\ ,\\ \frac{1}{3\kappa^2}\frac{\sqrt{V_{,ij}V^{,ij}}}{V} &= \frac{2}{3}\alpha^2\sqrt{\frac{(H_0+g_0)^2}{(H_0^2+g_0)^2}+\frac{16g_0(4\alpha-1)^2}{4g_0^2\alpha^2}\frac{1}{N^2}}\ll 1\ .\end{aligned} \tag{2.128}$$

Hence, we may choose conveniently the free parameters so that the slow-roll conditions are satisfied, and therefore, inflation takes place. From these expressions we see that the desired conditions will be obtained, in particular, when α is sufficiently small and/or H_0 and N are large enough. All these regimes help to fulfill the slow-roll conditions, in a quite natural way.

Cosmic acceleration with a pair of scalar fields

It is quite reasonable, and rather aesthetic, to think that the cosmic acceleration could be driven by the same mechanism as inflation. To this purpose, we apply the same model with two scalar fields, with the aim of reproducing late-time acceleration in a universe filled with a fluid with EoS $p_m=w_m\rho_m$. The free parameters given by the model could be adjusted to fit the observational data, as shown below. We begin with the action representing this model,

$$S=\int d^4x\sqrt{-g}\left[\frac{R}{2\kappa^2}-\frac{1}{2}(\partial\phi)^2-\frac{1}{2}(\partial\chi)^2-V(\phi,\chi)+L_m\right]\ . \tag{2.129}$$

By assuming a spatially flat FLRW metric, one obtains the Friedmann equations

$$H^2=\frac{\kappa^2}{3}\left[\frac{1}{2}(\dot\phi)^2+\frac{1}{2}(\dot\chi)^2+V(\phi,\chi)+\rho_m\right]\ ,\quad \dot H=-\frac{\kappa}{2}\left[\frac{1}{2}\dot\phi^2+\frac{1}{2}\dot\chi^2+V(\phi,\chi)+\rho_m\right]\ . \tag{2.130}$$

Variation of the action (2.129) yields the scalar field equations

$$\ddot\phi+3H\dot\phi+V_{,\phi}=0\ ,\quad \ddot\chi+3H\dot\chi+V_{,\chi}=0\ . \tag{2.131}$$

This set of independent equations may be supplemented by a fifth one, the conservation of matter-energy density, ρ_m ,

$$\dot{\rho_m}+3H\rho_m(1+w_m)=0\ . \tag{2.132}$$

As done in [216], we perform the substitutions

$$\Omega_m=\frac{\rho_m}{3H^2/\kappa^2}\ ,\quad \Omega_{sf}=\frac{\frac{1}{2}(\dot\phi)^2+\frac{1}{2}(\dot\chi)^2+V(\phi,\chi)}{3H^2/\kappa^2}\ ,\quad \epsilon=\frac{\dot H}{H^2}\ ,\quad w_{sf}\equiv\frac{p_{sf}}{\rho_{sf}}=\frac{\frac{1}{2}(\dot\phi)^2+\frac{1}{2}(\dot\chi)^2-V(\phi,\chi)}{\frac{1}{2}(\dot\phi)^2+\frac{1}{2}(\dot\chi)^2+V(\phi,\chi)}\ . \tag{2.133}$$

For convenience, we consider both scalar fields together under the subscript sf , so that the corresponding density parameter is $\Omega_{sf}=\Omega_\phi+\Omega_\chi$. Then, using the transformations (2.133), the Friedmann equations and the scalar equation read

$$\Omega_m+\Omega_{sf}=1\ ,\quad 2\epsilon+3(1+w_{sf})\Omega_{sf}+3(1+w_m)\Omega_m=0\ ,\quad \Omega_{sf}'+2\epsilon\Omega_{sf}+3\Omega_{sf}(1+w_{sf})=0\ . \tag{2.134}$$

Here, the prime denotes differentiation with respect to the number of e-folds $N \equiv \ln \frac{a(t)}{const}$. We can now combine Eqs. (2.133) to write the EoS parameter for the scalar fields as a function of Ω_{sf} and of the time derivative of the scalar fields, namely

$$w_{sf} = \frac{\kappa^2(\phi'^2 + \chi'^2) - 3\Omega_{sf}}{3\Omega_{sf}} \ . \tag{2.135}$$

Hence, it is possible to find an analytic solution for the equations (2.134), for a given evolution of the scalar fields, as will be seen below. Before doing that, it is useful to write the effective EoS parameter, which is given by

$$w_{\rm eff} = -1 - \frac{2\epsilon}{3} \ , \quad \rho_{\rm eff} = \rho_m + \rho_{sf} \ , \quad p_{\rm eff} = p_m + p_{sf} \ , \tag{2.136}$$

and the deceleration parameter

$$q = -\frac{\ddot{a}}{aH^2} = -1 - \epsilon \ . \tag{2.137}$$

As usual, for $q > 0$ the universe is in a decelerated phase, while $q < 0$ denotes an accelerated epoch, such that for $w_{\rm eff} < -1/3$ the expansion is accelerated. To solve the equations, we consider a universe which, at present, is filled with a pressureless component ($w_m = 0$) representing ordinary matter, and two scalar fields which represent a dynamical dark energy and a dark matter species. To show this, we make the following assumption on the evolution of the scalar fields, which are given as functions of N :

$$\phi(N) = \phi_0 + \frac{\alpha}{\kappa^2} N \ , \quad \chi(N) = \chi_0 + \frac{\beta}{\kappa^2} N \ . \tag{2.138}$$

Then, Eqs. (2.134) can be solved, and the scalar field density parameter takes the form

$$\Omega_{sf} = \Omega_\phi + \Omega_\chi = 1 - \frac{\lambda}{k\mathrm{e}^{\lambda N} + 3} \ , \tag{2.139}$$

where k is an integration constant and $\lambda = 3 - (\alpha^2 + \beta^2)$. It is possible to introduce an arbitrary function $g(N)$, to express the energy density parameter for each scalar field in the following way:

$$\Omega_\phi = 1 - \frac{\lambda}{k\mathrm{e}^{\lambda N} + 3} - g(N) \ , \quad \Omega_\chi = g(N) \ . \tag{2.140}$$

The function $g(N)$ may be chosen in such a way that the scalar field χ represents a cold dark matter contribution at present ($w_\chi \simeq 0$), and the scalar field ϕ represents the dark energy responsible for the accelerated expansion of our universe. On the other hand, using Eqs. (2.134), $\epsilon = \frac{H'}{H}$ is obtained as

$$\epsilon = -\frac{3}{2}\left\{1 - \frac{k\lambda\left(k\mathrm{e}^{\lambda N} + \alpha^2\beta^2\right)}{\left[(\alpha^2+\beta^2)\mathrm{e}^{-\lambda N} + 3k\right]\left(k\mathrm{e}^{\lambda N} + 3\right)}\right\} \ . \tag{2.141}$$

Then, it is possible to calculate the effective parameter of EoS given by Eq. (2.124)

$$w_{\rm eff} = -1 - \frac{2}{3}\epsilon = -\frac{k\lambda\left(k\mathrm{e}^{\lambda N} + \alpha^2\beta^2\right)}{\left[(\alpha^2+\beta^2)\mathrm{e}^{-\lambda N} + 3k\right]\left(k\mathrm{e}^{\lambda N} + 3\right)} \ . \tag{2.142}$$

We have four free parameters (N , k , α , β) that may be adjusted to fit the constraints derived from observations. With this purpose, we use the observational input $\Omega_m \simeq 0.03$, referred to baryonic matter, and normalize the number of e-folds N , taking $N = 0$ at present, then the integration constant k may be written as a function of α and β as

$$\Omega_m(N=0) = 0.03 \ \rightarrow \ k = \frac{2.01 - (\alpha^2+\beta^2)}{0.03} \ . \tag{2.143}$$

The free parameter β may be fixed in such a way that the scalar field χ represents cold dark matter at present, *i.e.*, $w_\chi \simeq 0$ and $\Omega_\chi \simeq 0.27$, and its EoS parameter is written as

$$w_\chi = \frac{\kappa^2 \chi'^2 - 3\Omega_\chi}{3\Omega_\chi} = \frac{\beta^2 - 3g(N)}{3g(N)} \ . \tag{2.144}$$

For convenience, we choose $g(N) = \frac{\beta^2}{3}\mathrm{e}^{-N}$, then the energy density and EoS parameter are given by

$$w_\chi = \frac{1 - \mathrm{e}^{-N}}{\mathrm{e}^{-N}} \ , \quad \Omega_\chi = \frac{\beta^2}{3}\mathrm{e}^{-N} \ . \tag{2.145}$$

Hence, at present ($N = 0$), the expressions (2.145) can be compared with the observational values and the β parameter is given by

$$w_\chi(N = 0) = 0 \ , \quad \Omega_\chi(N = 0) \simeq 0.27 \ \rightarrow \ \beta^2 = 0.81 \ . \tag{2.146}$$

Finally, the energy density of ϕ expressed by Eq. (2.140) takes the form

$$\Omega_\phi = 1 - \frac{\lambda}{k\mathrm{e}^{\lambda N} + 3} - \frac{\beta^2}{3}\mathrm{e}^{-N} \ . \tag{2.147}$$

The value for α could be taken so that $\Omega_\phi \simeq 0.7$ and $w_\phi \simeq -1$ at present. Hence, it has been shown that cosmic acceleration can be reproduced with a pair of scalar fields, where due to the presence of the extra scalar that can be identified with the dark matter component.

It is interesting to point out that one can unify these realistic descriptions of the inflationary and late-time acceleration eras within a single theory. However, the corresponding potential looks quite complicated. The easiest way would be to use step (θ-function) potentials in order to unify the whole description in the easiest way (as was pioneered in [105]).

We may also construct a model unifying early universe inflation and the present accelerated expansion era. To this end we can choose $f(\phi)$ in (2.28), which gives the Hubble parameter (2.29). Then, using (2.95), one can define $\omega(\phi)$ and $\sigma(\chi)$ with the help of an arbitrary function g. After defining then $f(\phi, \chi)$ with Eq. (2.92), we can construct the potential $V(\phi, \chi)$ using Eq. (2.93). Finally, we obtain the two scalar-tensor theory (2.88) reproducing the Hubble rate (2.29), which describes both inflation and the accelerated expansion.

2.6 Discussions

Modelling both early inflation and late-time acceleration within the context of a single theory has, undoubtedly, much aesthetic appeal and seems a worthy goal, which we attempted here. To summarize, we have developed the reconstruction program for the expansion history of the universe by using a single or multiple (canonic and/or phantom) scalar fields. Already in the case of single scalar, we have presented many examples which prove that it is possible to unify early-time inflation with late-time acceleration.

The reconstruction technique has then been generalized to the case of a scalar non-minimally coupled to the Ricci curvature, and to non-minimal (Brans-Dicke-type) scalars. Again, various explicit examples of unification of early-time inflation and late-time acceleration have been presented in such formulations. The chameleon mechanism is studied for our specific model, such that local constraints are satisfied

Finally, the case of several minimally coupled scalar fields has been considered in the description of the realistic evolution of the Hubble parameter, and we have shown that it is qualitatively easier to achieve the

realistic unification of late and early epochs in such a model in such a way as to satisfy the cosmological bounds coming from the observational data. We have shown that the perturbations could diverge in the case of a single scalar field that crosses the phantom barrier. This trouble is easily resolved by including a second scalar field, and imposing on each one a canonical (phantom) behavior. Using the freedom of choosing these scalar functions, one can constrain the theory in an observationally acceptable way. For instance, slow-roll conditions and stability conditions may be satisfied in different ways for different scalar functions, while the scale factor remains the same. This may be used also to obtain the correct perturbations structure, *etc.*

Chapter 3

Oscillating Universe from inhomogeneous EoS and coupled dark energy

[1]In the present chapter, we describe an oscillating Universe produced by an ideal dark fluid, what allows the possibility to unify early and late time acceleration under the same mechanism, in such a way that the Universe history may be reconstructed completely. On the other hand, it is important to keep in mind that these models represent just an effective description that owns a number of well-known problems, as the end of inflation. Nevertheless, they may represent a simple and natural way to resolve the coincidence problem, one of the possibilities may be an oscillating Universe (Refs. [11, 53, 149, 230, 307]), where the different phases of the Universe are reproduced due to its periodic behavior. The purpose of this chapter is to show that, from inhomogeneous EoS for a dark energy fluid, an oscillating Universe is obtained, and several examples are given to illustrate it. The possibility of an interaction between dark energy fluid, with homogeneous EoS, and matter is studied, which also reproduces that kind of periodic Hubble parameter, such case has been studied and is allowed by the observations (see [166, 165]). The possible phantom epochs are explored, and the possibility that the Universe may reach a Big Rip singularity (for a classification of future singularities, see Ref. [246]).

3.1 Inhomogeneous equation of state for dark energy

Let us consider firstly a Universe filled with a dark energy fluid, neglecting the rest of possible components (dust matter, radiation..), where its EoS depends on the Hubble parameter and its derivatives, such kind of EoS has been treated in several articles[47, 53, 75, 225, 229]. We show that for some choices of the EoS, an oscillating Universe resulted, which may include phantom phases. Then, the whole Universe history, from inflation to cosmic acceleration, is reproduced in such a way that observational constraints may be satisfied[179, 256]. We work in a spatially flat FLRW Universe. The Friedmann equations, considering

[1]This Chapter is based on the publication: [266].

now a perfect fluid, are obtained:

$$H^2 = \frac{\kappa^2}{3}\rho, \qquad \dot{H} = -\frac{\kappa^2}{2}(\rho + p) \ . \tag{3.1}$$

At this section, the EoS considered is given to have the general form:

$$p = w\rho + g(H, \dot{H}, \ddot{H}, ..; t) \ , \tag{3.2}$$

where w is a constant and $g(H, \dot{H}, \ddot{H}, ..; t)$ is an arbitrary function of the Hubble parameter H, its derivatives and the time t, (such kind of EoS has been treated in Ref. [225]). Using the FLRW equations (3.1) and (3.2), the following differential equation is obtained:

$$\dot{H} + \frac{3}{2}(1+w)H^2 + \frac{\kappa^2}{2}g(H, \dot{H}, \ddot{H}, ..; t) = 0 \ . \tag{3.3}$$

Hence, for a given function g, the Hubble parameter is calculated by solving the equation (3.3). It is possible to reproduce an oscillating Universe by an specific EoS (3.2). To illustrate this construction, let us consider the following g function as an example:

$$g(H, \dot{H}, \ddot{H}) = -\frac{2}{\kappa^2}\left(\ddot{H} + \dot{H} + \omega_0^2 H + \frac{3}{2}(1+w))H^2 - H_0\right) \ , \tag{3.4}$$

where H_0 and ω_0^2 are constants. By substituting (3.4) in (3.3) the Hubble parameter equation acquires the form:

$$\ddot{H} + \omega_0 H = H_0 \ , \tag{3.5}$$

which is the classical equation for an harmonic oscillator. The solution is found:

$$H(t) = \frac{H_0}{\omega_0^2} + H_1 \sin(\omega_0 t + \delta_0) \ , \tag{3.6}$$

where H_1 and δ_0 are integration constants. To study the system, we calculate the first derivative of the Hubble parameter, which is given by $\dot{H} = H_1 \cos(\omega_0 t + \delta_0)$, so the Universe governed by the dark energy fluid (3.4) oscillates between phantom and non-phantom phases with a frequency given by the constant ω_0, constructing inflation epoch and late-time acceleration under the same mechanism, and Big Rip singularity avoided

As another example, we consider the following EoS (3.2) for the dark energy fluid:

$$p = w\rho + \frac{2}{\kappa^2}Hf'(t) \ . \tag{3.7}$$

In this case $g(H;t) = \frac{2}{\kappa^2}Hf'(t)$, where $f(t)$ is an arbitrary function of the time t, and the prime denotes a derivative on t. The equation (3.3) takes the form:

$$\dot{H} + Hf'(t) = -\frac{3}{2}(1+w)H^2 \ . \tag{3.8}$$

This is the well-known Bernoulli differential equation. For a function $f(t) = -ln\,(H_1 + H_0 \sin\omega_0 t)$, where $H_1 > H_0$ are arbitrary constants, then the following solution for (3.8) is found:

$$H(t) = \frac{H_1 + H_0 \sin\omega_0 t}{\frac{3}{2}(1+w)t + k} \ , \tag{3.9}$$

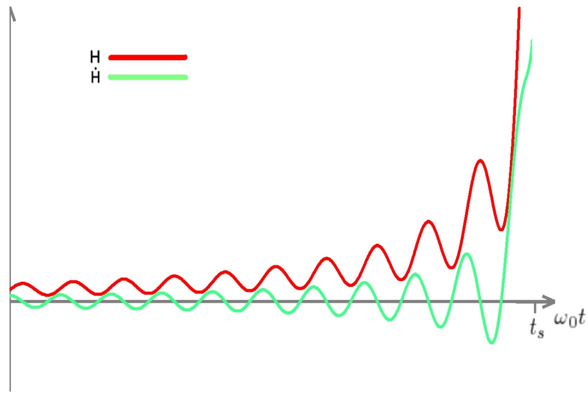

Figure 3.1: The Hubble parameter H and $\dot{H}$ for a value $w = -1.1$. Phantom phases occurs periodically, and a Big Rip singularity takes place at Rip time t_s.

here, the k is an integration constant. As it is seen, for some values of the free constant parameters, the Hubble parameter tends to infinity for a given finite value of t. The first derivative of the Hubble parameter is given by:

$$\dot{H} = \frac{\frac{H_0}{\omega_0}(\frac{3}{2}(1+w)t+k)\cos\omega_0 t - (H_1 + H_0\sin\omega_0 t)\frac{3}{2}(1+w)}{(\frac{3}{2}(1+w)t+k)^2} \ . \tag{3.10}$$

As it is shown in fig.3.1, the Universe has a periodic behavior, it passes through phantom and non-phantom epochs, with its respectives transitions. A Big Rip singularity may take place depending on the value of w, such that it is avoided for $w \geq -1$, while if $w < -1$ the Universe reaches the singularity in the Rip time given by $t_s = \frac{2k}{3|1+w|}$.

3.2 Dark energy ideal fluid and dust matter

3.2.1 No coupling between matter and dark energy

Let us now explore a more realistic model by introducing a matter component with EoS given by $p_m = w_m \rho_m$, we consider an inhomogeneous EoS for the dark energy component[75, 225]. It is shown below that an oscillating Universe may be obtained by constructing an specific EoS. In this case, the FLRW equations (3.1) take the form:

$$H^2 = -\frac{\kappa^2}{3}(\rho + \rho_m), \qquad \dot{H} = -\frac{\kappa^2}{2}\left(\rho + p + \rho_m + p_m\right) \ . \tag{3.11}$$

At this section, we consider a matter fluid that doesn't interact with the dark energy fluid, then the energy conservation equations are satisfied for each fluid separately:

$$\dot{\rho_m} + 3H(\rho_m + p_m) = 0, \qquad \dot{\rho} + 3H(\rho + p) = 0 \ . \tag{3.12}$$

It is useful to construct an specific solution for the Hubble parameter by defining the effective EoS with an effective parameter w_{eff}:

$$w_{eff} = \frac{p_{eff}}{\rho_{eff}}, \quad \rho_{eff} = \rho + \rho_m, \quad p_{eff} = p + p_m \ , \tag{3.13}$$

and the energy conservation equation $\dot{\rho}_{eff} + 3H(\rho_{eff} + p_{eff}) = 0$ is satisfied. We consider a dark energy fluid which is described by the EoS describes by the following expression:

$$p = -\rho + \frac{2}{\kappa^2}\frac{2(1+w(t))}{3\int(1+w(t))dt} - (1+w_m)\rho_{m0}\mathrm{e}^{-3(1+w_m)\int dt \frac{2}{3\int(1+w(t))}} , \tag{3.14}$$

here ρ_{m0} is a constant, and $w(t)$ is an arbitrary function of time t. Then the following solution is found:

$$H(t) = \frac{2}{3\int dt(1+w(t))} . \tag{3.15}$$

And the effective parameter (3.13) takes the form $w_{eff} = w(t)$. Then, it is shown that a solution for the Hubble parameter may be constructed from EoS (3.14) by specifying a function $w(t)$.

Let us consider an example[230] with the following function for $w(t)$:

$$w = -1 + w_0 \cos\omega t . \tag{3.16}$$

In this case, the EoS for the dark energy fluid, given by (3.14), takes the form:

$$p = -\rho + \frac{4}{3\kappa^2}\frac{\omega w_0\cos\omega t}{w_1 + w_0\sin\omega t} - (1+w_m)\rho_{m0}\mathrm{e}^{-3(1+w_m)\frac{2w}{3(w_1+w_0\sin\omega t)}} , \tag{3.17}$$

where w_1 is an integration constant. Then, by (3.15), the Hubble parameter yields:

$$H(t) = \frac{2\omega}{3(w_1 + w_0\sin\omega t)} . \tag{3.18}$$

The Universe passes through phantom and non phantom phases since the first derivative of the Hubble parameter has the form:

$$\dot{H} = -\frac{2\omega^2 w_0\cos\omega t}{3(w_1 + w_0\sin\omega t)^2} . \tag{3.19}$$

In this way, a Big Rip singularity will take place in order that $|w_1| < w_0$, and it is avoided when $|w_1| > w_0$. As it is shown, this model reproduces unified inflation and cosmic acceleration in a natural way, where the Universe presents a periodic behavior. In order to reproduce accelerated and decelerated phases, the acceleration parameter is studied, which is given by:

$$\frac{\ddot{a}}{a} = \frac{2\omega^2}{3(w_1 + w_0\sin\omega t)^2}\left(\frac{2}{3} - w_0\cos\omega t\right) . \tag{3.20}$$

Hence, if $w_0 > 2/3$ the different phases that Universe passes are reproduced by the EoS (3.17), presenting a periodic evolution that may unify all the epochs by the same description.

As a second example, we may consider a classical periodic function, the step function:

$$w(t) = -1 + \left\{ \begin{array}{ll} w_0 & 0 < t < T/2 \\ w_1 & T/2 < t < T \end{array} \right. , \tag{3.21}$$

and $w(t+T) = w(t)$. It is useful to use a Fourier expansion such that the function (3.21) become continuous. Approximating to third order, $w(t)$ is given by:

$$w(t) = -1 + \frac{(w_0 + w_1)}{2} + \frac{2(w_0 - w_1)}{\pi}\left(\sin\omega t + \frac{\sin 3\omega t}{3} + \frac{\sin 5\omega t}{5}\right) . \tag{3.22}$$

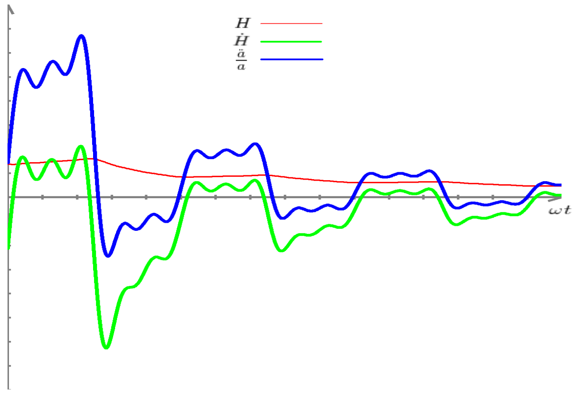

Figure 3.2: The Hubble parameter, its derivative and the acceleration parameter are represented for the "step" model for values $w_0 = -0.2$ and $w_1 = 1$.

Hence, the EoS for the Dark energy ideal fluid is given by (3.14), and the solution (3.15) takes the following form:

$$H(t) = \frac{2}{3}\left[w_2 + \frac{(w_0+w_1)}{2}t \right.$$
$$\left. -\frac{2(w_0-w_1)}{\pi\omega}\left(\cos\omega t + \frac{\cos 3\omega t}{9} + \frac{\cos 5\omega t}{25}\right)\right]^{-1} . \tag{3.23}$$

The model is studied by the first derivative of the Hubble parameter in order to see the possible phantom epochs, since:

$$\dot{H} = -\frac{3}{2}H^2\left[\frac{(w_0+w_1)}{2}\right.$$
$$\left. -\frac{2}{\pi}(w_0-w_1)\left(\sin\omega t + \frac{\sin 3\omega t}{3} + \frac{\sin 5\omega t}{5}\right)\right] . \tag{3.24}$$

Then, depending on the values from w_0 and w_1 the Universe passes through phantom phases. To explore the different epochs of acceleration and deceleration that the Universe passes on, the acceleration parameter is calculated:

$$\frac{\ddot{a}}{a} = H^2 + \dot{H} =$$
$$H^2\left[1 - \frac{3}{2}\left(\frac{(w_0+w_1)}{2} - \frac{2}{\pi}(w_0-w_1)\left(\sin\omega t + \frac{\sin 3\omega t}{3} + \frac{\sin 5\omega t}{5}\right)\right)\right] . \tag{3.25}$$

Then, in order to get acceleration and deceleration epochs, the constants parameters w_0 and w_1 may be chosen such that $w_0 < 2/3$ and $w_1 > 2/3$, as it is seen by (3.21). For this selection, phantom epochs take place in the case that $w_0 < 0$. In any case, the oscillated behavior is damped by the inverse term on the time t, as it is shown in fig.3.2, where the acceleration parameter is plotted for some given values of the free parameters. This inverse time term makes reduce the acceleration and the Hubble parameter such that the model tends to a static Universe.

We consider now a third example where a classical damped oscillator is shown, the function $w(t)$ is given by:

$$w(t) = -1 + \mathrm{e}^{-\alpha t} w_0 \cos\omega t , \tag{3.26}$$

here α and w_0 are two positive constants. Then, the EoS for the dark energy ideal fluid is constructed from (3.14). The solution for the Hubble parameter (3.15) is integrated, and takes the form:

$$H(t)=\frac{2}{3}\frac{\omega^2+\alpha^2}{w_1+w_0\mathrm{e}^{-\alpha t}(\omega\sin\omega t-\alpha\cos\omega t)}\ , \tag{3.27}$$

where w_1 is an integration constant. The Hubble parameter oscillates damped by an exponential term, and for big times, it tends to a constant $H(t\longrightarrow\infty)=\frac{2}{3}\frac{\omega^2+\alpha^2}{w_1}$, recovering the cosmological constant model. The Universe passes through different phases as it may be shown by the accelerated parameter:

$$\frac{\ddot{a}}{a}=H^2\left(1-\frac{3}{2}\mathrm{e}^{-\alpha t}w_0\cos\omega t\right)\ . \tag{3.28}$$

It is possible to restrict $w_0>\frac{2}{3}$ in order to get deceleration epochs when the matter component dominates. On the other hand, the Universe also passes through phantom epochs, since the Hubble parameter derivative gives:

$$\dot{H}=-\frac{3}{2}H^2\mathrm{e}^{-\alpha t}w_0\cos\omega t\ . \tag{3.29}$$

Hence, the example (3.26) exposes an oscillating Universe with a frequency given by ω and damped by a negative exponential term, which depends on the free parameter α, these may be adjusted such that the phases agree with the phases times constraints by the observational data.

3.2.2 Dark energy and coupled matter

In general, one may consider a Universe filled with a dark energy ideal fluid whose Eos is given $p=w\rho$, where w is a constant, and matter described by $p_m=\omega_m\rho_m$, both interacting with each other. In order to preserve the energy conservation, the equations for the energy density are written as following:

$$\dot{\rho_m}+3H(\rho_m+p_m)=Q,\qquad \dot{\rho}+3H(\rho+p_))=-Q\ , \tag{3.30}$$

here Q is an arbitrary function. In this way, the total energy conservation is satisfied $\dot{\rho}_{eff}+3H(\rho_{eff}+p_{eff})=0$, where $\rho_{eff}=\rho+\rho_m$ and $p_{eff}=p+p_m$, and the FLRW equations (3.11) doesn't change. To resolve this set of equations for a determined function Q, the second FLRW equation (3.11) is combined with the conservation equations (3.30), this yields:

$$\dot{H}=-\frac{\kappa^2}{2}\left[(1+w_m)\frac{\int Q\exp(\int dt3H(1+w_m))}{\exp(\int dt3H(1+w_m))}\right.$$
$$\left.+(1+w)\frac{-\int Q\exp(\int dt3H(1+w))}{\exp(\int dt3H(1+w))}\right]\ . \tag{3.31}$$

In general, this is difficult to resolve for a function Q. As a particular simple case is the cosmological constant where the dark energy EoS parameter $w=-1$ is considered, the equations become very clear, and (3.30) yields $\dot{\rho}=-Q$, which is resolved and the dark energy density is given by:

$$\rho(t)=\rho_0-\int dtQ(t)\ , \tag{3.32}$$

where ρ_0 is an integration constant. Then, the Hubble parameter is obtained by introducing (3.32) in the FLRW equations, which yields:

$$\dot{H}+\frac{3}{2}(1+w_m)H^2=\frac{\kappa^2}{2}(1+w_m)\left(\rho_0-\int dtQ\right)\ . \tag{3.33}$$

Hence, Hubble parameter depends essentially on the form of the coupling function Q. This means that a Universe model may be constructed from the coupling between matter and dark energy fluid, which is given by Q, an arbitrary function. It is shown below that some of the models given in the previous section by an inhomogeneous EoS dark energy fluid, are reproduced by a dark energy fluid with constant EoS ($w=-1$), but coupled to dust matter. By differentiating equation (3.33), the function Q may be written in terms of the Hubble parameter and its derivatives:

$$Q=-\frac{2}{\kappa^2}\frac{1}{1+w_m}\left(\ddot{H}+3(1+w_m)H\dot{H}\right) \ . \tag{3.34}$$

As an example, we use the solution(3.6):

$$H(t)=H_0+H_1\sin(\omega_0 t+\delta_0) \ . \tag{3.35}$$

Then, by the equation (3.33), the function Q is given by:

$$Q(t)=\frac{2}{\kappa^2(1+w_m)}\left[H_0\omega^2\sin\omega t+3(1+w_m)h_0\omega\cos\omega t(H_1+H_0\sin\omega t))\right] \ . \tag{3.36}$$

Then, the oscillated model (3.35) is reproduced by a coupling between matter and dark energy, which also oscillates. Some more complicated models may be constructed for complex functions Q. As an example let us consider the solution (3.18):

$$H(t)=\frac{2\omega}{3(w_1+w_0\sin\omega t)} \ . \tag{3.37}$$

The coupling function (3.34) takes the form:

$$Q(t)=-\frac{4}{3\kappa^2}\frac{\omega^3 w_0}{(1+w_m)(w_1+w_0\sin\omega t)^3}$$
$$\left[\sin\omega t(w_1+w_0\sin\omega t)^2+2w_0\cos^2\omega t-2(1+w_m)\cos\omega t\right] \ . \tag{3.38}$$

This coupling function reproduces a oscillated behavior that unifies the different epochs in the Universe. Hence, it have been shown that for a constant EoS for the dark energy with $w=-1$, inflation and late-time acceleration are given in a simple and natural way.

3.3 Scalar-tensor description

Let us now consider the solutions shown in the last sections through scalar-tensor description, such equivalence has been constructed in Ref. [81]. We assume, as before, a flat FLRW metric, a Universe filled with a ideal matter fluid with EoS given by $p_m=w_m\rho_m$, and no coupling between matter and the scalar field. Then, the following action is considered:

$$S=\int dx^4\sqrt{-g}\left[\frac{1}{2\kappa^2}R-\frac{1}{2}\omega(\phi)\partial_\mu\phi\partial^\mu\phi-V(\phi)+L_m\right] \ , \tag{3.39}$$

here $\omega(\phi)$ is the kinetic term and $V(\phi)$ represents the scalar potential. Then, the corresponding FLRW equations are written as:

$$H^2=\frac{\kappa^2}{3}\left(\rho_m+\rho_\phi\right) \ , \qquad \dot{H}=-\frac{\kappa^2}{2}\left(\rho_m+p_m+\rho_\phi+p_\phi\right) \ , \tag{3.40}$$

where ρ_ϕ and p_ϕ given by:

$$\rho_\phi = \frac{1}{2}\omega(\phi)\,\dot{\phi}^2 + V(\phi)\ , \qquad p_\phi = \frac{1}{2}\omega(\phi)\,\dot{\phi}^2 - V(\phi)\ . \tag{3.41}$$

By assuming:

$$\omega(\phi) = -\frac{2}{\kappa^2}f'(\phi) - (w_m+1)F_0\mathrm{e}^{-3(1+w_m)F(\phi)}\ ,$$
$$V(\phi) = \frac{1}{\kappa^2}\left[3f(\phi)^2 + f'(\phi)\right] + \frac{w_m-1}{2}F_0\,\mathrm{e}^{-3(1+w_m)F(\phi)}\ . \tag{3.42}$$

The following solution is found:

$$\phi = t\ , \quad H(t) = f(t)\ , \tag{3.43}$$

which yields:

$$a(t) = a_0\mathrm{e}^{F(t)}, \qquad a_0 = \left(\frac{\rho_{m0}}{F_0}\right)^{\frac{1}{3(1+w_m)}}\ . \tag{3.44}$$

Then, we may assume solution (3.27), in such case the $f(\phi)$ function takes the form:

$$f(\phi) = \frac{2}{3}\frac{\omega^2+\alpha^2}{w_1 + w_0\mathrm{e}^{-\alpha\phi}(\omega\sin\omega\phi - \alpha\cos\omega\phi)}\ , \tag{3.45}$$

And by (3.42) the kinetic term and the scalar potential are given by:

$$\omega(\phi) = \frac{3}{\kappa^2}f^2(\phi)w_0\mathrm{e}^{-\alpha\phi}\cos\omega\phi - (1+w_m)F_0\mathrm{e}^{-3(1+w_m)F(\phi)},$$
$$V(\phi) = \frac{3f^2(\phi)}{\kappa^2}\left(1 - \frac{1}{2}w_0\mathrm{e}^{-\alpha\phi}\cos\omega\phi\right) + \frac{w_m-1}{2}F_0\mathrm{e}^{-3(1+w_m)F(\phi)}, \tag{3.46}$$

where $F(\phi) = \int d\phi f(\phi)$ and F_0 is an integration constant. Then, the periodic solution (3.27) is reproduced in the mathematical equivalent formulation in scalar-tensor theories by the action (3.39) and explicit kinetic term and scalar potential, in this case, is given by (3.46).

3.4 Discussions

In this chapter, a Universe model has been presented that reproduces in a natural way the early and late-time acceleration by a periodic behavior of the Hubble parameter. The late-time transitions are described by this model: the transition from deceleration to acceleration, and the possible transition from non-phantom to phantom epoch. The observational data does not restrict yet the nature and details of the EoS for dark energy, then the possibility that Universe behaves periodically is allowed. For that purpose, several examples have been studied in the present chapter, some of them driven by an inhomogeneous EoS for dark energy, which just represents an effective description of nature of dark energy, and others by a coupling between dark energy and matter which also may provide another possible constraint to look for.

Part II

On modified theories of gravity and its implications in Cosmology

Chapter 4

Modified f(R) gravity from scalar-tensor theories and inhomogeneous EoS dark energy

[1]In this chapter modified f(R) theories of gravity are introduced in the context of cosmology which avoid the need to introduce dark energy, and may give an explanation about the origin of the current accelerated expansion and even on the expansion history of the Universe (for recent reviews and books on the topic, see [78, 240, 281]).
In this sense, the cosmic acceleration and the cosmological properties of metric formulation f(R) theories have been studied in Refs.[1, 7, 29, 44, 60, 35, 61, 74, 76, 79, 83, 91, 94, 95, 99, 110, 106, 107, 119, 144, 122, 148, 196, 204, 209, 212, 227, 219, 221, 245, 248, 263, 305]. Recently the main focus has been improviding a f(R)-theory that reproduces the whole history of the Universe, including the early accelerated epoch (inflation) and the late-time acceleration at the current epoch, Refs. [105, 228, 234, 236, 237], where the possible future singularities have been studied in the context of f(R)-gravity(see Ref. [236]). It is important to remark that the main problem that this kind of theories found at the begining of its development was the local gravitational test; nowadays several viable models have been proposed, which pass the solar system tests and reproduce the cosmological history (see [10, 88, 176, 228, 234, 235, 236, 237, 258, 286]).
The reconstruction of f(R)-gravity is shown, to be possible in the cosmological context by using an auxiliary scalar field and then various examples are given where the current accelerated expansion is reproduced and also the whole history of the Universe. Also the analogy between the so-called dark fluids, whose EoS is inhomogeneous and which have been investigated as effective descriptions of dark energy in Chapter 3, and the f(R)-theories is investigated.

[1]This Chapter is based on the publication: [265].

4.1 Reconstruction of f(R)-gravity

In this section, it will be shown how f(R) theory may be reconstructed in such a way that cosmological solutions can be obtained. Let us start with the action for f(R)-gravity:

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left(f(R) + L_m \right) \ . \tag{4.1}$$

Here L_m denotes the lagrangian of some kind of matter. The field equations are obtained by varying the action on $g_{\mu\nu}$, then they are given by:

$$R_{\mu\nu} f'(R) - \frac{1}{2} g_{\mu\nu} f(R) + g_{\mu\nu} \Box f'(R) - \nabla_\mu \nabla_\nu f'(R) = \kappa^2 T^{(m)}_{\mu\nu} \ , \tag{4.2}$$

where $T^{(m)}_{\mu\nu}$ is the energy-momentum tensor for the matter that filled the Universe. We assume a flat FLRW metric:

$$ds^2 = -dt^2 + a^2(t) \sum_{i=1}^{3} dx_i^2 \ . \tag{4.3}$$

Then, if $T^{(m)}_{\mu\nu}$ is a perfect fluid, the modified Friedmann equations for the Hubble parameter $H(t) = \frac{\dot{a}}{a}$, take the form:

$$\begin{aligned} \frac{1}{2} f(R) - 3(H^2 + \dot{H}) f'(R) + 18 f''(R)(H^2 \dot{H} + H \ddot{H}) &= \kappa^2 \rho_m \ , \\ \frac{1}{2} f(R) - (3H^2 + \dot{H}) f'(R) - \Box f'(R) &= -\kappa^2 p_m \ , \end{aligned} \tag{4.4}$$

where the Ricci scalar is given by $R = 6(2H^2 + \dot{H})$. Hence, by the equations (4.4), any cosmology may be reproduced for a given function $f(R)$. Nevertheless, in general it is very difficult to get an exact cosmological solution directly from (4.4). It is a very useful technique developed in [219, 228], where an auxiliary scalar field without kinetic term is introduced, then the action (4.1) is rewritten as follows:

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left(P(\phi) R + Q(\phi) \right) \ , \tag{4.5}$$

where the scalar field ϕ has no kinetic term. By variation on the metric tensor $g_{\mu\nu}$ the field equation is obtained:

$$\frac{1}{2} g_{\mu\nu} \left(P(\phi) R + Q(\phi) \right) + P(\phi) R_{\mu\nu} + g_{\mu\nu} \Box P(\phi) - \nabla_\mu \nabla_\nu P(\phi) = \kappa^2 T^{(m)}_{\mu\nu} \ . \tag{4.6}$$

The action (4.5) gives an additional equation for the scalar field ϕ, obtained directly from the action by varying it with respect to ϕ:

$$P'(\phi) R + Q'(\phi) = 0 \ , \tag{4.7}$$

here the primes denote derivatives respect ϕ. This equation may be resolved with the scalar field as a function of R, $\phi = \phi(R)$, and then, replacing this result in the action (4.5), the action (4.1) is recovered,

$$f(R) = P\left(\phi(R)\right) R + Q\left(\phi(R)\right) \ . \tag{4.8}$$

Hence, any cosmological model could be solved by the field equation (4.6), and then by (4.7) and (4.8) the function $f(R)$ is obtained. For the metric (4.3), the Friedmann equations read:

$$\begin{aligned} 3H \frac{dP(\phi)}{dt} + 3H^2 P(\phi) + \frac{1}{2} Q(\phi) - \kappa^2 \rho_m &= 0 \ , \\ \frac{d^2 P(\phi)}{dt^2} + 2H \frac{dP(\phi)}{dt} + (2\dot{H} + 3H^2) P(\phi) + \kappa^2 p &= 0 \ . \end{aligned} \tag{4.9}$$

We redefine the scalar field such that it is chosen to be the time coordinate $\phi = t$. The perfect fluid define by the energy-momentum tensor $T^{(m)}_{\mu\nu}$ may be seen as a sum of the different components (radiation, cold dark matter,..) which filled our Universe and whose equation of state (EoS) is given by $p_m = w_m \rho_m$, then by the energy momentum conservation $\dot{\rho}_m + 3H(1+w_m)\rho_m = 0$, it gives:

$$\rho_m = \rho_{m0} \exp\left(-3(1+w_m)\int dt H(t)\right) . \tag{4.10}$$

Hence, taking into account the equations (4.9) and (4.10) the Hubble parameter may be calculated as a function of the scalar field ϕ, $H = g(\phi)$. By combining the equations (4.9), the function $Q(\phi)$ is deleted, and it yields:

$$2\frac{d^2P(\phi)}{d\phi^2} - 2g(\phi)\frac{dP(\phi)}{d\phi} + 4g'(\phi)P(\phi) + (1+w_m)\exp\left[-3(1+w_m)\int d\phi g(\phi)\right] = 0 . \tag{4.11}$$

By resolving this equation for a given function $P(\phi)$, a cosmological solution $H(t)$ is found, and the function $Q(\phi)$ is obtained by means of the equation given by (4.9):

$$Q(\phi) = -6(g(\phi))^2 P(\phi) - 6g(\phi)\frac{dP(\phi)}{dt} . \tag{4.12}$$

If we neglect the contribution of matter, then the equation (4.11) is a first order differential equation on $g(\phi)$, and it can be easily resolved. The solution found is the following:

$$g(\phi) = -\sqrt{P(\phi)}\int d\phi \frac{P''(\phi)}{2P^{2/3}(\phi)} + kP(\phi) , \tag{4.13}$$

where k is an integration constant. As an example to show this construction, let us choose the following function that, as it is shown below, reproduce late-time acceleration:

$$P(\phi) = \phi^{\alpha} , \quad \text{where} \quad \alpha > 1 . \tag{4.14}$$

Then, by the result (4.13), the following solution is found:

$$g(\phi) = k\phi^{\alpha/2} + \frac{\alpha(\alpha-1)}{\alpha+2}\frac{1}{\phi} , \tag{4.15}$$

where k is an integration constant. By the expression (4.12), the function $Q(\phi)$ is given by:

$$Q(\phi) = -6\left[(k\phi)^2 + \left(k + \frac{\alpha(2\alpha+1)}{\alpha+2}\right)\phi^{\frac{3\alpha-2}{2}} + \frac{\alpha^2(\alpha-1)(2\alpha+1)}{(\alpha+2)^2}\phi^{\alpha-2}\right] . \tag{4.16}$$

The function (4.15) gives the following expression for the Hubble parameter:

$$H(t) = kt^{\alpha/2} + \frac{\alpha(\alpha-1)}{\alpha+2}\frac{1}{t} . \tag{4.17}$$

This solution may reproduce a Universe that passes through two phases for a conveniently choice of α. For small times the Hubble and the acceleration parameter take the expressions:

$$\begin{aligned} H(t) &\sim \frac{\alpha(\alpha-1)}{\alpha+2}\frac{1}{t} , \\ \frac{\ddot{a}}{a} &\sim -\frac{\alpha(\alpha-1)}{\alpha+2}\left(1 - \frac{\alpha(\alpha-1)}{\alpha+2}\right)\frac{1}{t^2} , \end{aligned} \tag{4.18}$$

where if $1+\sqrt{3} < \alpha \leq 2$, the Universe is in a decelerated epoch for small times, which may be interpreted as the radiation/matter dominated epochs. When t is large, the Hubble parameter takes the form:

$$H(t) = kt^{\alpha/2} \ . \tag{4.19}$$

This clearly gives an accelerated expansion that coincides with the current expansion that Universe experiences nowadays. Finnally, the expression for $f(R)$ (4.8) is calculated by means of (4.14) and (4.16), and by the expression of the Ricci scalar $R = 6(2g^2(\phi) + g(\phi))$, which is used to get $\phi(R)$. For simplicity, we study the case where $\alpha = 2$, which gives:

$$\phi = \sqrt{\frac{R - 2k \pm \sqrt{R(1-2k)}}{2}} \ . \tag{4.20}$$

By inserting this expression into (4.14) and (4.16), the function $f(R)$ is obtained:

$$f(R) = [R - 6(k(k+1) + 5/2)] \, \frac{R - 2k \pm \sqrt{R(1-2k)}}{2} + const \ . \tag{4.21}$$

Thus, with this expression for the function $f(R)$, the current cosmic acceleration is reproduced with the solution (4.17). In general, as it is seen in the following example, it is very difficult to reconstruct the function $f(R)$ for the whole expansion history, and even more difficult for the kind of models that unify inflation and cosmic acceleration, in this cases it is convenient to study the asymptotic behaviour of the model, and then by resolving the equations, the expression for $f(R)$ is obtained .

As a second example, one could try to reconstruct the whole Universe history, from inflation to cosmic acceleration by the f(R)-gravity. In this case, we proceed in the inverse way than above, by suggesting a function $g(\phi)$, and trying to reconstruct the expression $f(R)$ by calculating $P(\phi)$ and $Q(\phi)$ by means of equations (4.9). We study an example suggested in the above chapter, where:

$$g(\phi) = \frac{H_1}{\phi^2} + \frac{H_0}{t_s - \phi} \ . \tag{4.22}$$

For this function the Hubble parameter takes the form $H(t) = \frac{H_1}{t^2} + \frac{H_0}{t_s - t}$. To reconstruct the form of $f(R)$ we have to resolve the equation (4.11). For simplicity we study the assimptotically behaviour of $H(t)$ in such a way that allow us to resolve easily the equation (4.11) for $P(\phi)$. Then, for small t ($t << t_s$), the Hubble and acceleration parameters read:

$$H(t) \sim \frac{H_1}{t^2} \quad \frac{\ddot{a}}{a} \sim \frac{H_1}{t^2}\left(\frac{H_1}{t^2} - \frac{2H_1}{t} + \frac{2H_0}{t_s - t}\right) \ . \tag{4.23}$$

As it is observed, for t close to zero, $\frac{\ddot{a}}{a} > 0$, so the Universe is in an accelerated epoch during some time, which may be interpreted as the inflation epoch, and for $t > 1/2$ ($t << t_s$), the Universe enters in a decelerated phase, interpreted as the radiation/matter dominated epoch. The equation (4.11) for $P(\phi)$, neglecting the matter component, is given by:

$$\frac{d^2P(\phi)}{d\phi^2} - \frac{H_1}{\phi^2}\frac{dP(\phi)}{d\phi} - \frac{4H_1}{\phi^3}P(\phi) = 0 \ . \tag{4.24}$$

The solution of (4.24) is:

$$P(\phi) = k\phi^{-4} + \frac{20k}{H_1}\phi^{-3} + \frac{120k}{H_1^2}\phi^{-2} + \frac{240k}{H_1^3}\phi^{-1} + \frac{120k}{H_1^4} \ , \tag{4.25}$$

where k is an integration constant. The function $Q(\phi)$ given by (4.12), takes the form:

$$Q(\phi) = -6\frac{H_1}{\phi^2}\left[kH_1\phi^{-6} + 6k\phi^{-4} - \frac{120k}{H_1^3}\phi^{-2}\right] . \tag{4.26}$$

By using the expression for the Ricci scalar, the relation $\phi(R) \sim (12H_1^2/R)^{1/4}$ is found, then, the function $f(R)$ for small values of ϕ is approximately:

$$f(R) \sim \frac{k}{24H_1}R^2 . \tag{4.27}$$

Hence, by the expression (4.27) the early cosmological behaviour of the Universe (4.23), where a first accelerated epoch (inflation) occurs and after it, a decelerated phase comes (radiation/matter dominated epochs), is reproduced. Let us now investigate the large values for t (t close to the Rip time t_s). In this case the Hubble and the acceleration parameters for the solution (4.22) take the form:

$$g(\phi) \sim \frac{H_0}{t_s - \phi} \quad \to H(t) \sim \frac{H_0}{t_s - t} \quad \text{and} \quad \frac{\ddot{a}}{a} \sim \frac{H_0(H_0+1)}{(t_s - t)^2} . \tag{4.28}$$

As it is observed, for large t the solution (4.22) gives an accelerated epoch which enters in a phantom phase ($\dot{H} > 0$) and ends in a Big Rip singularity at $t = t_s$. In this case the equation for $P(\phi)$ reads:

$$\frac{d^2P(\phi)}{d\phi^2} - \frac{H_0}{t_s - \phi}\frac{dP(\phi)}{d\phi} + \frac{2H_0}{(t_s - \phi)^2}P(\phi) = 0 . \tag{4.29}$$

The possible solutions of the equation (4.29) depend on the value of the constant H_0 , as it follows:

1. If $H_0 > 5 + 2\sqrt{6}$ or $H_0 < 5 - 2\sqrt{6}$, then the following solution for $P(\phi)$ is found:

$$P(\phi) = A(t_s - \phi)^{\alpha_+} + B(t_s - \phi)^{\alpha_-},$$
$$\text{where} \quad \alpha_\pm = \frac{H_0 + 1 \pm \sqrt{H_0(H_0+10)+1}}{2} , \tag{4.30}$$

   Then, through the expression (4.12), the function $Q(\phi)$ is calculated. In this case, for t close to t_s , the Ricci scalar takes the form $R = \frac{6H_0(2H_0+1)}{(t_s-t)^2}$, and hence it takes large values, diverging at the Rip time $t = t_s$. The function $f(R)$, for a large R , takes the form:

$$f(R) \sim R^{1-\alpha_-/2} . \tag{4.31}$$

2. If $5 - 2\sqrt{6} < H_0 < 5 + 2\sqrt{6}$, the solution of (4.29) is given by:

$$P(\phi) = (t_s - \phi)^{-(h_0+1)/2}\left[A\cos\left((t_s - \phi)\ln\frac{-H_0^2 + 10H_0 - 1}{2}\right)\right.$$
$$\left.+B\sin\left((t_s - \phi)\ln\frac{-H_0^2 + 10H_0 - 1}{2}\right)\right] . \tag{4.32}$$

   Then, for this choice of the constant H_0 , and by means of the equation (4.8) the form of the function $f(R)$ is found:

$$f(R) \sim R^{(H_0+1)/4}\left[A\cos\left(R^{-1/2}\ln\frac{-H_0^2 + 10H_0 - 1}{2}\right)\right.$$
$$\left.+B\sin\left(R^{-1/2}\ln\frac{-H_0^2 + 10H_0 - 1}{2}\right)\right] . \tag{4.33}$$

Hence, the expressions (4.31) and (4.33) for $f(R)$ reproduce the behaviour of the Hubble parameter for large t given in (4.28), where a phantom accelerated epoch ocurrs, and the Universe ends in a Big Rip singularity for $t = t_s$. As is shown, this model is reproduced by (4.27) for small t when the curvature R is large, and by (4.31) or (4.33) when t is large. For a proper choice of the power α, the solution for large t is given by (4.31), which in combination with the solution (4.27) for small t, it looks like standard gravity $f(R) \sim R$ for intermediate t. On the other hand, for negavite powers in (4.31) and in combination with (4.27), this takes a similar form than the model suggested in [219], $f(R) \sim R + R^2 + 1/R$, which is known that passes qualitatively most of the solar system tests. As this is an approximated form, it is reasonable to think that this model follows from some non-linear gravity of the sort Ref.[237], which may behave as R^2 for large R. The stability of this kind of models (for a detailed discussion see Ref. [6]), whose solutions are given by (4.27) and (4.33), is studied in Ref. [228], where the transition between epochs is well done, and then, the viable cosmological evolution may be reproduced by these models. The quantitative study of the transition between different cosmological epochs will be analyzed later.

Suming up it has been shown that any cosmology may be reproduced by $f(R)$-gravity by using an auxiliary scalar field and resolving the equations (4.11) and (4.12) to reconstruct such function of the Ricci scalar. To fix the free parameters in the theory, it would be convenient to contrast the model with the observational data as the supernova data by means of the evolution of the scale parameter which is obtained in the models shown above.

4.2 F(R)-gravity and dark fluids

In this section the mathematical equivalence between f(R) theories, that could reproduce a given cosmology as it was seen above, and the standard cosmology with a dark fluid included whose EoS has inhomogeneous terms that depend on the Hubble parameter and its derivatives, is investigated. Let us start with the modified Friedmann equations (4.4) written in the following form:

$$\begin{aligned} 3H^2 &= \frac{1}{f'(R)}\left(\frac{1}{2}f(R) - \partial_{tt}f'(R) - \Box f'(R)\right) - 3\dot{H} \ , \\ -3H^2 - 2\dot{H} &= \frac{1}{f'(R)}\left(\Box f'(R) - \frac{1}{2}f(R) + H\partial_t f'(R)\right) - \dot{H} \ , \end{aligned} \tag{4.34}$$

where we have neglected the contributions of any other kind of matter. If we compare Eqs. (4.34) with the standard Friedmann equations ($3H^2 = \kappa^2\rho$ and $-3H^2 - 2\dot{H} = \kappa^2 p$), we may identify both right sides of Eqs. (4.34) with the energy and pressure densities of a perfect fluid, in such a way that they are given by:

$$\begin{aligned} \rho &= \frac{1}{\kappa^2}\left[\frac{1}{f'(R)}\left(\frac{1}{2}f(R) - \partial_{tt}f'(R) - \Box f'(R)\right) - 3\dot{H}\right] \\ p &= \frac{1}{\kappa^2}\left[\frac{1}{f'(R)}\left(\Box f'(R) - \frac{1}{2}f(R) + H\partial_t f'(R)\right) - \dot{H}\right] \ . \end{aligned} \tag{4.35}$$

Then, Eqs. (4.34) take the form of the usual Friedmann equations, where the parameter of the EoS for this dark fluid is defined by:

$$w = \frac{p}{\rho} = \frac{\frac{1}{f'(R)}\left(\Box f'(R) - \frac{1}{2}f(R) + H\partial_t f'(R)\right) - \dot{H}}{\frac{1}{f'(R)}\left(\frac{1}{2}f(R) - \partial_{tt}f'(R) - \Box f'(R)\right) - 3\dot{H}} \ . \tag{4.36}$$

And the corresponding EoS may be written as follows:

$$p = -\rho - \frac{1}{\kappa^2}\left(4\dot{H} + \frac{1}{f'(R)}\partial_{tt}f'(R) - \frac{H}{f'(R)}\partial_t f'(R)\right) \ . \tag{4.37}$$

The Ricci scalar is a function given by $R = 6(2H^2 + \dot{H})$, then $f(R)$ is a function on the Hubble parameter H and its derivative $\dot{H}$. The inhomogeneus EoS for this dark fluid (4.37) takes the form of the kind of dark fluids studying in several works (see [47, 53, 75, 225, 229]), and particularly the form of the EoS for dark fluids investigated in Chapter 3, which is written as follows:

$$p = -\rho + g(H, \dot{H}, \ddot{H}...), \quad \text{where}$$

$$g(H, \dot{H}, \ddot{H}...) = -\frac{1}{\kappa^2}\left(4\dot{H} + \partial_{tt}(\ln f'(R)) + (\partial_t \ln f'(R))^2 - H\partial_t \ln f'(R)\right) . \tag{4.38}$$

Then, as constructed in Chapter 3, by combining the Friedmann equations, it yields the following differential equation:

$$\dot{H} + \frac{\kappa^2}{2} g(H, \dot{H}, \ddot{H}...) = 0 . \tag{4.39}$$

Hence, for a given cosmological model, the function g given in (4.38) may be seen as a function of cosmic time t, and then by the time-dependence of the Ricci scalar, the function g is rewritten in terms of R. Finnally, the function $f(R)$ is recovered by the expression (4.38). In this sense, Eq. (4.39) combining with the expression (4.38) results in:

$$\frac{dx(t)}{dt} + x(t)^2 - H(t)x(t) = \dot{H}(t) , \tag{4.40}$$

where $x(t) = \frac{d(lnf'(R(t))}{dt}$. Eq. (4.40) is a type of Riccati equation, that may be solved by a given Hubble parameter. Hence, $f(R)$-gravity is constructed from standard cosmology where a perfect fluid with an inhomogeneus EoS is included. To show this, let us consider a simple example:

$$H(t) = \frac{H_1}{t} , \quad H_1, H_0 > 0 . \tag{4.41}$$

This model describes a power-law solution, very comoon in cosmology as it can reproduces radiation/matter epochs as well as accelerating expansion. Then, by inserting (4.41) inb the differential equation (4.40), and after some calculations, the solution yields,

$$f(R) = \kappa_1^2 R^{1-\frac{H_1+1+\sqrt{1+H_1(10+H_1)}}{4}} + \lambda , \tag{4.42}$$

where κ_1^2 is a constant that depends on H_1 while λ is an integration constant. Then, by the function (4.42) the model (4.41) is reproduced. The analog dark fluid that reproduces this behaviour, may be found by inserting the function (4.42) in the EoS for the fluid given by (4.38). Then, it could be studied as an effective fluid with its evolution and compared with other models. However, note that for a general example, the non-linear equation (4.40) has to be resolved numerically, and just simple examples can be analytycally solved. Thus, as it is shown, f(R)-gravity may be written as dark fluids with particular dependence on the Hubble parameter and its derivatives through its EoS (4.38), so the same model may be interpretated in several ways.

4.3 Discussions

f(R)-gravity theories may provide an alternative description of the current accelerated epoch of our Universe and even on the whole expansion history. As it is pointed in several works, one may construct this kind of theories in accordance with the local test of gravity and with the observational data, which provide that, at the current epoch, the effective parameter of the EoS is close to -1. The next step should be to compare the different cosmological tests, as the supernovae luminosity distance or the positions of the

CMB peaks, with the F(R) models. On the other hand, we have shown two different ways of reconstruct the f(R)-gravity in the context of cosmology, in the first one, an auxiliary scalar field is used, and in the second one, the mathematical equivalence between f(R)-gravity and dark fluids with inhomogeneus EoS shows that while the expansion history of the Universe may be interpreted as a perfect fluid whose EoS has dependence on the cosmological evolution, this effect may be caused by the modification of the classical theory of gravity. However, there is not any complementary probe to distinguish between both descriptions of the evolution of the Universe, and thus, such kind of modified gravity is completly allowed. Hence, f(R)-gravity is an acceptable solution to the cosmological problem, that may provide new interesting constraints to look for. In the next chapters, the reproduction of Λ CDM model will be studied in the context of modified theories of gravity, and even transition to phantom epochs.

Chapter 5

Cosmological reconstruction of realistic F(R) gravities

[1]In the preceeding chapter, it was explored the reconstruction of cosmological solutions in $F(R)$ gravity by means of auxiliary fields. It turns out that in most cases this reconstruction is done in the presence of the auxiliary scalar which may be excluded at the final step so that any FLRW cosmology may be realized within specific reconstructed $F(R)$ gravity. However, the weak point of a so developed reconstruction scheme is that the final function $F(R)$ represents usually some polynomial in positive/negative powers of scalar curvature. On the same time, the viable models have strongly non-linear structure.

In the present chapter we develop the new scheme for cosmological reconstruction of $F(R)$ gravity in terms of e-folding (or, redshift z) so that there is no need to use more complicated formulation with auxiliary scalar [18, 19, 20, 83, 93, 228, 233]. Using such technique the number of examples are presented where $F(R)$ gravity is reconstructed so that it gives the well-known cosmological evolution: Λ CDM epoch, deceleration/acceleration epoch which is equivalent to presence of phantom and non-phantom matter, late-time acceleration with the crossing of phantom-divide line, transient phantom epoch and oscillating universe. It is shown that some generalization of such technique for viable $F(R)$ gravity is possible, so that local tests are usually satisfied. In this way, modified gravity unifying inflation, radiation/matter dominance and dark energy eras may be further reconstructed in the early or in the late universe so that the future evolution may be different. This opens the way to non-linear reconstruction of realistic $F(R)$ gravity. Moreover, it is demonstrated that cosmological reconstruction of viable modified gravity may help in the formulation of non-singular models in finite-time future. The reconstruction suggests the way to change some cosmological predictions of the theory in the past or in the future so that it becomes easier to pass the available observational data. Finally, we show that our method works also for modified gravity with scalar theory and any requested cosmology may be realized within such theory too.

[1]This Chapter is based on the publications: [126, 242]

5.1 Cosmological reconstruction of modified $F(R)$ gravity

From the starting action of $F(R)$ gravity, the field equation corresponding to the first FLRW equation is:

$$0 = -\frac{F(R)}{2} + 3\left(H^2 + \dot{H}\right) F'(R) - 18(\left(4H^2\dot{H} + H\ddot{H}\right) F''(R) + \kappa^2\rho \,. \tag{5.1}$$

with $R = 6\dot{H} + 12H^2$. We now rewrite Eq.(5.1) by using a new variable (which is often called e-folding) instead of the cosmological time t, $N = \ln\frac{a}{a_0}$. The variable N is related with the redshift z by $e^{-N} = \frac{a_0}{a} = 1 + z$. Since $\frac{d}{dt} = H\frac{d}{dN}$ and therefore $\frac{d^2}{dt^2} = H^2\frac{d^2}{dN^2} + H\frac{dH}{dN}\frac{d}{dN}$, one can rewrite (5.1) by

$$0 = -\frac{F(R)}{2} + 3\left(H^2 + HH'\right) F'(R) - 18(\left(4H^3H' + H^2\left(H'\right)^2 + H^3H''\right) F''(R) + \kappa^2\rho \,. \tag{5.2}$$

Here $H' \equiv dH/dN$ and $H'' \equiv d^2H/dN^2$. If the matter energy density ρ is given by a sum of the fluid densities with constant EoS parameter w_i, we find

$$\rho = \sum_i \rho_{i0} a^{-3(1+w_i)} = \sum_i \rho_{i0} a_0^{-3(1+w_i)} \mathrm{e}^{-3(1+w_i)N} \,. \tag{5.3}$$

Let the Hubble rate is given in terms of N via the function $g(N)$ as

$$H = g(N) = g\left(-\ln\left(1+z\right)\right) \,. \tag{5.4}$$

Then scalar curvature takes the form: $R = 6g'(N)g(N) + 12g(N)^2$, which could be solved with respect to N as $N = N(R)$. Then by using (5.3) and (5.4), one can rewrite (5.2) as

$$\begin{aligned} 0 \;=\;& -18\left(4g\left(N\left(R\right)\right)^3 g'\left(N\left(R\right)\right) + g\left(N\left(R\right)\right)^2 g'\left(N\left(R\right)\right)^2 + g\left(N\left(R\right)\right)^3 g''\left(N\left(R\right)\right)\right)\frac{d^2F(R)}{dR^2} \\ & +3\left(g\left(N\left(R\right)\right)^2 + g'\left(N\left(R\right)\right) g\left(N\left(R\right)\right)\right)\frac{dF(R)}{dR} - \frac{F(R)}{2} + \sum_i \rho_{i0} a_0^{-3(1+w_i)} \mathrm{e}^{-3(1+w_i)N(R)} \,, \end{aligned} \tag{5.5}$$

which constitutes a differential equation for $F(R)$, where the variable is scalar curvature R. Instead of g, if we use $G(N) \equiv g\left(N\right)^2 = H^2$, the expression (5.5) could be a little bit simplified:

$$\begin{aligned} 0 \;=\;& -9G\left(N\left(R\right)\right)\left(4G'\left(N\left(R\right)\right) + G''\left(N\left(R\right)\right)\right)\frac{d^2F(R)}{dR^2} + \left(3G\left(N\left(R\right)\right) + \frac{3}{2}G'\left(N\left(R\right)\right)\right)\frac{dF(R)}{dR} \\ & -\frac{F(R)}{2} + \sum_i \rho_{i0} a_0^{-3(1+w_i)} \mathrm{e}^{-3(1+w_i)N(R)} \,. \end{aligned} \tag{5.6}$$

Note that the scalar curvature is given by $R = 3G'(N) + 12G(N)$. Hence, when we find $F(R)$ satisfying the differential equation (5.5) or (5.6), such $F(R)$ theory admits the solution (5.4). Hence, such $F(R)$ gravity realizes above cosmological solution.

5.2 ΛCDM model in F(R) gravity

Let us reconstruct the $F(R)$ gravity which reproduces the ΛCDM-era. In the Einstein gravity, the FLRW equation for the ΛCDM cosmology is given by

$$\frac{3}{\kappa^2}H^2 = \frac{3}{\kappa^2}H_0^2 + \rho_0 a^{-3} = \frac{3}{\kappa^2}H_0^2 + \rho_0 a_0^{-3}\mathrm{e}^{-3N} \,. \tag{5.7}$$

Here H_0 and ρ_0 are constants. The first term in the r.h.s. corresponds to the cosmological constant and the second term to the cold dark matter (CDM). The (effective) cosmological constant Λ in the present universe is given by $\Lambda = 3H_0^2$. Then one gets

$$G(N) = H_0^2 + \frac{\kappa^2}{3}\rho_0 a_0^{-3} \mathrm{e}^{-3N} \ , \tag{5.8}$$

and $R = 3G'(N) + 12G(N) = 12H_0^2 + \kappa^2\rho_0 a_0^{-3}\mathrm{e}^{-3N}$, which can be solved with respect to N as follows,

$$N = -\frac{1}{3}\ln\left(\frac{\left(R - 12H_0^2\right)}{\kappa^2\rho_0 a_0^{-3}}\right) \ . \tag{5.9}$$

By considering the homogeneous part of Eq.(5.6), it takes the following form:

$$0 = 3\left(R - 9H_0^2\right)\left(R - 12H_0^2\right)\frac{d^2F(R)}{d^2R} - \left(\frac{1}{2}R - 9H_0^2\right)\frac{dF(R)}{dR} - \frac{1}{2}F(R) \ . \tag{5.10}$$

By changing the variable from R to x by $x = \frac{R}{3H_0^2} - 3$, Eq.(5.10) reduces to the hypergeometric differential equation:

$$0 = x(1-x)\frac{d^2F}{dx^2} + \left(\gamma - (\alpha+\beta+1)\,x\right)\frac{dF}{dx} - \alpha\beta F \ . \tag{5.11}$$

Here

$$\gamma = -\frac{1}{2} \ , \alpha + \beta = -\frac{1}{6} \ , \quad \alpha\beta = -\frac{1}{6} \ , \tag{5.12}$$

Solution of (5.11) is given by Gauss' hypergeometric function $F(\alpha, \beta, \gamma; x)$:

$$F(x) = AF(\alpha, \beta, \gamma; x) + Bx^{1-\gamma}F(\alpha - \gamma + 1, \beta - \gamma + 1, 2 - \gamma; x) \ . \tag{5.13}$$

Here A and B are constant. Thus, we demonstrated that modified $F(R)$ gravity may describe the Λ CDM epoch without the need to introduce the effective cosmological constant. However, a deeper study on the solution (5.13) is necessary. Let us write the scale factor a in terms of the Ricci scalar through out (5.9), it yields

$$a(R) = \left(\frac{\kappa^2\rho_0}{R - 4\Lambda}\right)^{(1/3)} \ , \tag{5.14}$$

where $\Lambda = 3H_0^2$. In order to ensure the positivity of the scale factor, the Ricci scalar has to be restricted to $R \geq 4\Lambda$. We can see that the equation (5.10) has two singular points at $R = 3\Lambda$ and $R = 4\Lambda$. From the allowed range for the Ricci scalar, we have that one of the poles is out of the range while the other one is located at the boundary. Nevertheless, the argument of the solution (5.13), given by $x = \frac{R}{\Lambda} - 3$, ensure the convergency of the solution when $|x| \leq 1$, what restricts the value of the Ricci scalar to $R \leq 4\Lambda$, otherwise the function is either divergent or complex valued. Them, in order to ensure a real and finite gravitational action, we have to choose the integration constants $A = B = 0$. Hence, we have to consider just the particular solution coming from the inhomogeneous part of the equation (5.7), which will depend on the kind of fluid presenced in the Universe. By considering dustlike matter ($w = 0$), and substituting in the Friedmann equation (5.7), we get the particular solution,

$$F(R) = R - 2\Lambda \ , \tag{5.15}$$

which is the well known action for General Relativity. Then, the only physical solution for a model trying to reproduce a Hubble parameter behaving as Λ CDM solution turns to be the Einstein-Hilbert action with positive cosmological constant. Nevertheless, other approximate Λ CDM solutions can be well reproduced in $F(R)$ gravity as it is shown along the pesent work.

5.3 Reconstruction of approximate ΛCDM solutions

As an another example, we reconstruct $F(R)$ gravity reproducing the system with non-phantom matter and phantom matter in the Einstein gravity, whose FLRW equation is given by

$$\frac{3}{\kappa^2}H^2 = \rho_q a^{-c} + \rho_p a^c \ . \tag{5.16}$$

Here ρ_q , ρ_p , and c are positive constants. When a is small as in the early universe, the first term in the r.h.s. dominates and it behaves as the universe described by the Einstein gravity with a matter whose EoS parameter is $w = -1 + c/3 > -1$, that is, non-phantom like. On the other hand, when a is large as in the late universe, the second term dominates and behaves as a phantom-like matter with $w = -1 - c/3 < -1$. Then since $G(N) \equiv g\left(N\right)^2 = H^2$, we find

$$G = G_q \mathrm{e}^{-cN} + G_p \mathrm{e}^{cN} \ , \quad G_q \equiv \frac{\kappa^2}{3}\rho_q a_0^{-c} \ , \qquad G_p \equiv \frac{\kappa^2}{3}\rho_p a_0^{c} \ . \tag{5.17}$$

Then since $R = 3G'(N) + 12G(N)$,

$$\mathrm{e}^{cN} = \frac{R \pm \sqrt{R^2 - 4\left(144 - 9c^2\right)}}{2\left(12 + 3c\right)} \ , \tag{5.18}$$

when $c \neq 4$ and

$$\mathrm{e}^{cN} = \frac{R}{24G_p} \ , \tag{5.19}$$

when $c = 4$. In the following, just for simplicity, we consider $c = 4$ case. In the case, the non-phantom matter corresponding to the first term in the r.h.s. of (5.16) could be radiation with $w = 1/3$. Then Eq.(5.6) in this case is given by

$$0 = -6\left(\frac{24G_pG_q}{R} + \frac{R}{24}\right) R\frac{d^2F(R)}{dR^2} + \frac{9}{2}\left(-\frac{24G_pG_q}{R} + \frac{R}{24}\right)\frac{dF(R)}{dR} - \frac{F(R)}{2} \ . \tag{5.20}$$

By changing variable R to x by $R^2 = -576G_pG_qx$, we can rewrite Eq.(5.20) as

$$0 = (1-x)x\frac{d^2F}{dx^2} + \left(\frac{3}{4} + \frac{x}{4}\right)\frac{dF}{dx} - \frac{F}{2} \ , \tag{5.21}$$

whose solutions are again given by Gauss' hypergeometric function (5.13) with

$$\gamma = \frac{3}{4} \ , \quad \alpha + \beta + 1 = -\frac{1}{4} \ , \quad \alpha\beta = \frac{1}{2} \ . \tag{5.22}$$

Let us now study a model where the dominant component is phantom-like one. Such kind of system can be easily expressed in the standard General Relativity when a phantom fluid is considered, where the FLRW equation reads $H^2(t) = \frac{\kappa^2}{3}\rho_{\rm ph}$, Here the subscript ph denotes the phantom nature of the fluid. As the EoS for the fluid is given by $p_{\rm ph} = w_{\rm ph}\rho_{\rm ph}$ with $w_{\rm ph} < -1$, by using the conservation equation $\dot{\rho}_{\rm ph} + 3H(1+w_{\rm ph})\rho_{\rm ph} = 0$, the solution for the FLRW equation $H^2(t) = \frac{\kappa^2}{3}\rho_{\rm ph}$ is well known, and it yields $a(t) = a_0(t_s - t)^{-H_0}$, where a_0 is a constant, $H_0 = -\frac{1}{3(1+w_{\rm ph})}$ and t_s is the so-called Rip time. Then, the solution describes the Universe that ends at the Big Rip singularity in the time t_s . The same behavior can be achieved in $F(R)$ theory with no need to introduce a phantom fluid. The equation (5.6) can be solved and the expression for the $F(R)$ that reproduces the solution is reconstructed. The expression for

the Hubble parameter as a function of the number of e-folds is given by $H^2(N) = H_0^2 e^{2N/H_0}$. Then, the equation (5.6), with no matter contribution, takes the form:

$$R^2 \frac{d^2F(R)}{dR^2} + AR\frac{dF(R)}{dR} + BF(R) = 0 \ , \tag{5.23}$$

where $A = -H_0(1 + H_0)$ and $B = \frac{(1+2H_0)}{2}$. This equation is the well known Euler equation whose solution yields

$$F(R) = C_1 R^{m_+} + C_2 R^{m_-} \ , \quad \text{where} \quad m_\pm = \frac{1 - A \pm \sqrt{(A-1)^2 - 4B}}{2} \ . \tag{5.24}$$

Thus, the phantom dark energy cosmology $a(t) = a_0(t_s - t)^{-H_0}$ can be also obtained in the frame of $F(R)$ theory and no phantom fluid is needed.

We can consider now the model where the transition to the phantom epoch occurs. It has been pointed out that $F(R)$ could behave as an effective cosmological constant, such that its current observed value is well reproduced. One can reconstruct the model where late-time acceleration is reproduced by an effective cosmological constant and then the phantom barrier is crossed. Such transition, which may take place at current time, could be achieved in $F(R)$ gravity. The solution considered can be expressed as:

$$H^2 = H_1 \left(\frac{a}{a_0}\right)^m + H_0 = H_1 e^{mN} + H_0 \ , \tag{5.25}$$

where H_1 , H_0 and α are positive constants. This solution can be constructed in GR when a cosmological constant and a phantom fluid are included. In the present case, the solution (5.25) can be achieved just by an $F(R)$ function, such that the transition from non-phantom to phantom epoch is reproduced. Scalar curvature can be written in terms of the number of e-folds again. Then, the equation (5.6) takes the form:

$$x(1-x)F''(x) + \left[x\left(-\frac{6+m}{6m}\right) - \frac{1}{3m}\right]F'(x) - \frac{m+4}{m}F(x) = 0 \ , \tag{5.26}$$

where $x = \frac{1}{3H_0(m+4)}(12H_0 - R)$. The equation (5.26) reduces to the hypergeometric differential equation (5.13), so the solution is given, as in some of the examples studied above, by the Gauss' hypergeometric function (5.16), whose parameters for this case are given by

$$\gamma = -\frac{1}{3m} \ , \quad \alpha + \beta = -\frac{3m+2}{2m} \ , \quad \alpha\beta = \frac{m+4}{2m} \ , \tag{5.27}$$

and the obtained $F(R)$ gravity produces the FLRW cosmology with the late-time crossing of the phantom barrier in the universe evolution.

Another example with transient phantom behavior in $F(R)$ gravity can be achieved by following the same reconstruction described above. In this case, we consider the following Hubble parameter:

$$H^2(N) = H_0 \ln\left(\frac{a}{a_0}\right) + H_1 = H_0 N + H_1 \ , \tag{5.28}$$

where H_0 and H_1 are positive constants. For this model, we have a contribution of an effective cosmological constant, and a term that will produce a superaccelerating phase although no future singularity will take place(compare with earlier model [1] with transient phantom era). The solution for the model (5.28) can be expressed as a function of time

$$H(t) = \frac{a_0 H_0}{2}(t - t_0) \tag{5.29}$$

Then, the Universe is superaccelerating, but as it can be seen from (5.29), in spite of its phantom nature, no future singularity occurs. The differential reconstruction equation can be obtained as

$$a_2 x \frac{d^2F(x)}{dx} + (a_1 x + b_1)\frac{dF(x)}{dx} + b_0 F(x) = 0 \ , \tag{5.30}$$

where we have performed a variable change $x = H_0 N + H_1$, and the constant parameters are $a_2 = H_0^2$, $a_1 = -H_0$ $b_1 = -\frac{H_0^2}{2}$ and $b_0 = 2H_0$. The equation (5.30) is a kind of the degenerate hypergeometric equation, whose solutions are given by the Kummers series $K(a, b; x)$:

$$F(R) = K\left(-2, -\frac{1}{2H_0}; \frac{R - 3H_0}{12}\right) \ . \tag{5.31}$$

Hence, such $F(R)$ gravity has cosmological solution with the transient phantom behavior which does not evolve to future singularity.

Let us now consider the case where a future contracting Universe is reconstructed in this kind of models. We study a model where the universe is currently accelerating, then the future contraction of the Universe occurs. The following solution for the Hubble parameter is considered,

$$H(t) = 2H_1(t_0 - t) \ , \tag{5.32}$$

where H_1 and t_0 are positive constants. For this example, the Hubble parameter (5.32) turns negative for $t > t_0$, when the Universe starts to contract itself, while for $t \ll t_0$, the cosmology is typically Λ CDM one. Using notations $\tilde{H_0} = 4H_1 t_0^2$ and $\tilde{H_1} = 4H_1$ and repeating the above calculation, one gets:

$$F(R) = K\left(-8\tilde{H}_1, -\frac{\tilde{H}_1}{8}; \frac{12\tilde{H}_0 - 3\tilde{H}_1 - R}{12\tilde{H}_1}\right) \ . \tag{5.33}$$

Hence, the oscillating cosmology (5.32) that describes the asymptotically contracting Universe with a current accelerated epoch can be found in specific $F(R)$ gravity.

Thus, we explicitly demonstrated that $F(R)$ gravity reconstruction is possible for any cosmology under consideration without the need to introduce the auxiliary scalar. However, the obtained modified gravity has typically polynomial structure with terms which contain positive and negative powers of curvature as in the first such model unifying the early-time inflation and late-time acceleration [219, 223]. As a rule such models do not pass all the local gravitational tests. Some generalization of above cosmological reconstruction is necessary.

5.3.1 Cosmological solutions in $f(R)$ gravity with the presence of an inhomogeneous EoS fluid

We consider now a Universe governed by some specific $f(R)$ theory in the presence of a perfect fluid, whose equation of state is given by,

$$p = w(a)\rho + \zeta(a) \ , \tag{5.34}$$

where $w(a)$ and $\zeta(a)$ are functions of the scale factor a , which could correspond to the dynamical behavior of the fluid and to its viscosity. Let us write the FLRW equations for f(R) as following,

$$H^2 = \frac{1}{3}(\rho' + \rho_{f(R)}) \ ; , \tag{5.35}$$

$$2\dot{H} + 3H^2 = -(p' + p_{f(R)}) \ , \tag{5.36}$$

where $\rho' = \frac{\rho}{f'(R)}$ and $p' = \frac{p}{f'(R)}$. The pressure and energy density with the subscript $f(R)$ contains the terms corresponded to $f(R)$ and are defined as

$$\rho_{f(R)} = \frac{1}{f'}\left(\frac{Rf'-f}{2} - 3H\dot{R}f''\right) , \tag{5.37}$$

$$p_{f(R)} = \frac{1}{f'}\left(\dot{R}^2 f''' + 2H\dot{R}f'' + \ddot{R}f'' + \frac{1}{2}(f - Rf')\right) . \tag{5.38}$$

Then, by combining both FLRW equations, and using the equation of state defined in (5.34), we can write

$$\zeta(a) = \left(w(a)\rho_{f(R)} - p_{f(R)} - 2\dot{H} - 3(1+w(a))H^2\right) f'(R(a)) . \tag{5.39}$$

As $\zeta(a)$ just depends on the Hubble parameter and its derivatives, for some specific solutions, any kind of cosmology can be reproduced. Let us consider the example,

$$\frac{3}{\kappa^2}H^2 = G_\rho a^{-c} + G_q a^d , \tag{5.40}$$

where G_ρ and G_q are constants. We can check that in this solution, the first term in the r.h.s. corresponds to a fluid with EoS $w_p = -1+c/3 > -1$, while the second term, it has an equation of state $w_q = -1-d/3 < -1$, which corresponds to a phantom fluid. We could consider a viable $f(R)$ function proposed in [237], which is given by

$$F(R) = \frac{\alpha R^{m+l} - \beta R^n}{1+\gamma R^l}. \tag{5.41}$$

This function is known to pass the local gravity tests and could contribute to drive the Universe to an accelerated phase. Then, by introducing (5.40) and (5.41) in the expression for $\zeta(a)$ in (5.39), we obtain the equation of state for the inhomogeneous fluid that, together with $f(R)$, reproduces the solution (5.40), which for $c = 3$ and $d \geq 0$ reproduces the Λ CDM model, and probably drives the Universe evolution into a phantom phase in the near future.

5.4 Cosmological evolution of viable $F(R)$ gravity

In this section, we show how the cosmological reconstruction may be applied to viable modified gravity which passes the local gravitational tests. In this way, the non-linear structure of modified $F(R)$ gravity may be accounted for, unlike the previous section where only polynomial $F(R)$ structures may be reconstructed. Let us write $F(R)$ in the following form: $F(R) = F_0(R) + F_1(R)$. Here we choose $F_0(R)$ as a known function like that of GR or one of viable $F(R)$ models introduced in [10, 88, 176, 196], or viable $F(R)$ theories unifying inflation with dark energy [105, 234], for example

$$F_0(R) = \frac{1}{2\kappa^2}\left(R - \frac{(R-R_0)^{2n+1} + R_0^{2n+1}}{f_0 + f_1\left\{(R-R_0)^{2n+1} + R_0^{2n+1}\right\}}\right) . \tag{5.42}$$

Using the procedure similar to the one of second section, one gets the reconstruction equation corre-

sponding to (5.6)

$$\begin{aligned}
0 \;=\; & -9G\left(N\left(R\right)\right)\left(4G'\left(N\left(R\right)\right)+G''\left(N\left(R\right)\right)\right)\frac{d^2F_0(R)}{dR^2}+\left(3G\left(N\left(R\right)\right)+\frac{3}{2}G'\left(N\left(R\right)\right)\right)\frac{dF_0(R)}{dR} \\
& -\frac{F_0(R)}{2} \\
& -9G\left(N\left(R\right)\right)\left(4G'\left(N\left(R\right)\right)+G''\left(N\left(R\right)\right)\right)\frac{d^2F_1(R)}{dR^2}+\left(3G\left(N\left(R\right)\right)+\frac{3}{2}G'\left(N\left(R\right)\right)\right)\frac{dF_1(R)}{dR} \\
& -\frac{F_1(R)}{2}+\sum_i \rho_{i0}a_0^{-3(1+w_i)}\mathrm{e}^{-3(1+w_i)N(R)} \ . \qquad (5.43)
\end{aligned}$$

The above equation can be regarded as a differential equation for $F_1(R)$. For a given $G(N)$ or $g(N)$ (5.4), if one can solve (5.6)) as $F(R)=\hat{F}(R)$, we also find the solution of (5.43) as $F_1(R)=\hat{F}(R)-F_0(R)$. For example, for $G(N)$ (5.8), by using (5.13), we find

$$F_1(R)=AF(\alpha,\beta,\gamma;x)+Bx^{1-\gamma}F(\alpha-\gamma+1,\beta-\gamma+1,2-\gamma;x)-F_0(R) \ . \qquad (5.44)$$

Here α, β, γ, and x are given by $x=\frac{R}{3H_0^2}-3$ and (5.12). Using $F_0(R)$ (5.42) one has

$$F_1(R)=AF(\alpha,\beta,\gamma;x)+Bx^{1-\gamma}F(\alpha-\gamma+1,\beta-\gamma+1,2-\gamma;x)-\frac{1}{2\kappa^2}\left(R-\frac{\left(R-R_0\right)^{2n+1}+R_0^{2n+1}}{f_0+f_1\left\{\left(R-R_0\right)^{2n+1}+R_0^{2n+1}\right\}}\right) , \qquad (5.45)$$

which describes the asymptotically de Sitter universe. Instead of $x=\frac{R}{3H_0^2}-3$ and (5.12), if we choose α, β, γ, and x as $R^2=-576G_pG_qx$ and in (5.22), $F_1(R)$ (5.45) shows the asymptotically phantom universe behavior, where H diverges in future.

One may start from $F_0(R)$ given by hypergeometric function (5.13) with $x=\frac{1}{3H_0(m+4)}(12H_0-R)$ and (5.27). In such a model, there occurs Big Rip singularity. Let $\tilde{F}(R)$ be $F(R)$ again given by hypergeometric function (5.13) with $x=\frac{R}{3H_0^2}-3$ and (5.12):

$$\begin{aligned}
&\tilde{F}(R)=\tilde{A}F(\tilde{\alpha},\tilde{\beta},\tilde{\gamma};\tilde{x})+\tilde{B}\tilde{x}^{1-\tilde{\gamma}}F(\tilde{\alpha}-\tilde{\gamma}+1,\tilde{\beta}-\tilde{\gamma}+1,2-\tilde{\gamma};\tilde{x}) , \\
&\tilde{x}=\frac{R}{3H_0^2}-3 , \quad \tilde{\gamma}=-\frac{1}{2} , \tilde{\alpha}+\tilde{\beta}=-\frac{1}{6} , \quad \tilde{\alpha}\tilde{\beta}=-\frac{1}{6} \ . \qquad (5.46)
\end{aligned}$$

If we choose $F(R)=\tilde{F}(R)$, the Λ CDM model emerges. Then choosing $F_1(R)=\tilde{F}(R)-F_0(R)$, the Big Rip singularity, which occurs in $F_0(R)$ model, does not appear and the universe becomes asymptotically de Sitter space. Hence, the reconstruction method suggests the way to create the non-singular modified gravity models [1, 18, 19, 20, 77, 123, 193, 236, 272]. Of course, it should be checked that reconstruction term is not large (or it affects only the very early-time/late-time universe) so that the theory passes the local tests as it was before the adding of correction term.

Gauss' hypergeometric function $F(\alpha,\beta,\gamma;x)$ is defined by

$$F(\alpha,\beta,\gamma;x)=\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)}\sum_{n=0}^{\infty}\frac{\Gamma(\alpha+n)\Gamma(\beta+n)}{\Gamma(\gamma+n)n!}z^n \ . \qquad (5.47)$$

Since

$$\alpha_0,\beta_0=\frac{-3m-2\pm\sqrt{m^2-20m+4}}{4m}<0 , \quad \tilde{\alpha},\tilde{\beta}=\frac{-1\pm 5}{12} , \qquad (5.48)$$

when R is large, $F_1(R)$ behaves as $F_1(R) \sim R^{\left(3m+2+\sqrt{m^2-20m+4}\right)/4m}$. In spite of the above expression, since the total $F(R) = F_0(R) + F_1(R)$ is given by $\tilde{F}(R)$ (5.46), the Big Rip type singularity does not occur. The asymptotic behavior of $F_1(R)$ cancels the large R behavior in $F_0(R)$ suggesting the way to present the non-singular cosmological evolution.

We now consider the case that H and therefore G oscillate as

$$G(N) = G_0 + G_1 \sin\left(\frac{N}{N_0}\right) , \tag{5.49}$$

with positive constants G_0, G_1, and N_0. Let the amplitude of the oscillation is small but the frequency is large:

$$G_0 \gg \frac{G_1}{N_0} , \quad N_0 \gg 1 . \tag{5.50}$$

When $G_1 = 0$, we obtain de Sitter space, where the scalar curvature is a constant $R = 12G_0$. Writing $G(N)$ as

$$G = \frac{R}{6} - G_0 , \tag{5.51}$$

by using (5.6), one arrives at general relativity:

$$F(R) = c_0 \left(R - 6G_0\right) . \tag{5.52}$$

Instead of (5.51), using an arbitrary function $\tilde{F}$, if we write

$$G = G_0 + \tilde{F}(R) - \tilde{F}(12G_0) , \tag{5.53}$$

we obtain a general $F(R)$ gravity, which admits de Sitter space solution. When $G_1 \neq 0$, under the assumption (5.50), one may identify $F(R)$ in (5.52) with $F_0(R)$. We now write $G(N)$ and the scalar curvature R as

$$G(N) = \frac{R}{6} - G_0 + \frac{G_1}{N_0} g(N) , \quad R = 12G_0 + \frac{3G_1}{N_0} r(N) , \tag{5.54}$$

with adequate functions $g(N)$ and $r(N)$. Then since $R = 6g'(N)g(N) + 12g(N)^2$ and from (5.50), we find

$$g(N) = -\left(N_0 \sin\frac{N}{N_0} + \frac{1}{2}\cos\frac{N}{N_0}\right) , \quad r(N) = 4N_0 \sin\frac{N}{N_0} + \cos\frac{N}{N_0} \tag{5.55}$$

By assuming

$$F(R) = c_0 \left(R - 6G_0 + \frac{G_1^2}{N_0^3} f(R)\right) , \tag{5.56}$$

and identifying

$$F_1(R) = \frac{c_0 G_1^2}{N_0^3} f(R) , \tag{5.57}$$

from (5.43), one obtains

$$0 = G_0 \frac{df}{dr} - \sin\left(\frac{N}{N_0}\right) + o\left(\frac{G_1}{N_0}, N_0\right) , \tag{5.58}$$

which can be solved as

$$f(R) = -\frac{1}{2G_0}\left(\cos^{-1} r \mp r\sqrt{1-r^2}\right) . \tag{5.59}$$

Then at least perturbatively, one can construct a model which exhibits the oscillation of H.

Before going further, let us find $F(R)$ equivalent to the Einstein gravity with a perfect fluid with a constant EoS parameter w, where H behaves as

$$\frac{3}{\kappa^2}H^2 = \rho_0 \mathrm{e}^{-3(w+1)} . \tag{5.60}$$

Then

$$G(N) = \frac{\kappa^2\rho_0}{3}\mathrm{e}^{-3(w+1)} , \quad R(N) = (1-3w)\,\kappa^2\rho_0\mathrm{e}^{-3(w+1)} , \tag{5.61}$$

which could be solved as

$$N = -\frac{1}{3(w+1)}\ln\frac{R}{(1-3w)\,\kappa^2\rho_0} . \tag{5.62}$$

Therefore Eq.(5.6) has the following form:

$$0 = \frac{3(1+w)}{1-3w}R^2\frac{d^2F(R)}{dR^2} - \frac{1+3w}{2(1-3w)}R\frac{dF(R)}{dR} - \frac{F(R)}{2} , \tag{5.63}$$

whose solutions are given by a sum of powers of R

$$F(R) = F_+R^{n_+} + F_-R^{n_-} . \tag{5.64}$$

Here $F_\pm$ are constants of integration and $n_\pm$ are given by

$$n_\pm = \frac{1}{2}\left\{\frac{7+9w}{6(1+w)} \pm \sqrt{\left(\frac{7+9w}{6(1+w)}\right)^2 + \frac{2(1-3w)}{3(1+w)}}\right\} . \tag{5.65}$$

If $w > -1/3$, the universe is decelerating but if $-1 < w < -1/3$, the universe is accelerating as in the quintessence scenario.

By using the solution (5.13), which mimics Λ CDM model, and the solution (5.64), one may consider the following model:

$$F(x) = \left\{AF(\alpha,\beta,\gamma;x) + Bx^{1-\gamma}F(\alpha-\gamma+1,\beta-\gamma+1,2-\gamma;x)\right\}\frac{\mathrm{e}^{\lambda\left(\frac{R}{R_1}-\frac{R_1}{R}\right)}}{\mathrm{e}^{\lambda\left(\frac{R}{R_1}-\frac{R_1}{R}\right)}+\mathrm{e}^{-\lambda\left(\frac{R}{R_1}-\frac{R_1}{R}\right)}} + F_+R^{n_+} + F_-R^{n_-} . \tag{5.66}$$

Here R_1 is a constant which is sufficiently small compared with the curvature R_0 in the present universe. On the other hand, we choose a positive constant λ to be large enough. We also choose $F_\pm$ to be small enough so that only the first term dominates when $R \gg R_1$. Note that the factor $\frac{\mathrm{e}^{\lambda\left(\frac{R}{R_1}-\frac{R_1}{R}\right)}}{\mathrm{e}^{\lambda\left(\frac{R}{R_1}-\frac{R_1}{R}\right)}+\mathrm{e}^{-\lambda\left(\frac{R}{R_1}-\frac{R_1}{R}\right)}}$ behaves as step function when λ is large:

$$\lim_{\lambda\to+\infty}\frac{\mathrm{e}^{\lambda\left(\frac{R}{R_1}-\frac{R_1}{R}\right)}}{\mathrm{e}^{\lambda\left(\frac{R}{R_1}-\frac{R_1}{R}\right)}+\mathrm{e}^{-\lambda\left(\frac{R}{R_1}-\frac{R_1}{R}\right)}} = \theta(R-R_1) \equiv \left\{\begin{array}{ll} 1 & \text{when } R > R_1 \\ 0 & \text{when } R < R_1 \end{array}\right. . \tag{5.67}$$

Then in the early universe and in the present universe, only the first term dominates and the Λ CDM universe could be reproduced. In the future universe where $R \ll R_1$, the factor $\frac{\mathrm{e}^{\lambda\left(\frac{R}{R_1}-1\right)}}{\mathrm{e}^{\lambda\left(\frac{R}{R_1}-1\right)}+\mathrm{e}^{-\lambda\left(\frac{R}{R_1}-1\right)}}$ decreases very rapidly and the second terms in (5.66) dominate. Then if $w > -1/3$, the universe decelerates again but if $-1 < w < -1/3$, the universe will be accelerating as in the quintessence scenario.

Thus, we explicitly demonstrated that the viable $F(R)$ gravity may be reconstructed so that any requested cosmology may be realized after the reconstruction. Moreover, one can use the viable $F(R)$

gravity unifying the early-time inflation with late-time acceleration (and manifesting the radiation/matter dominance era between accelerations) and passing local tests in such a scheme. The (small) correction term $F_1(R)$ can be always constructed so that it slightly corrects (if necessary) the cosmological bounds being relevant only at the very early/late universe. This scenario opens the way to extremely realistic description of the universe evolution in $F(R)$ gravity consistent with local tests and cosmological bounds.

5.5 Discussion

In summary, we developed a general scheme for cosmological reconstruction of modified $F(R)$ gravity in terms of e-folding (or redshift) without use of auxiliary scalar in intermediate calculations. Using this method, it is possible to construct the specific modified gravity which contains any requested FLRW cosmology. The number of $F(R)$ gravity examples is found where the following background evolution may be realized: Λ CDM epoch, deceleration with subsequent transition to effective phantom superacceleration leading to Big Rip singularity, deceleration with transition to transient phantom phase without future singularity, oscillating universe. It is important that all these cosmologies may be realized only by modified gravity without use of any dark components (cosmological constant, phantom, quintessence, etc). In particular we find that the only real valued gravitational action that reproduces an exact Λ CDM expansion is the Hilbert-Einstein action with a positive cosmological constant. However, it has also shown that approximate solutions that mimic Λ CDM model at the current time, can well be reproduced in terms of a $F(R)$ function.

It is also shown that our method may be applied to viable $F(R)$ gravities which pass local tests and unify the early-time inflation with late-time acceleration. In this case, the additional reconstruction may be made so that correction term is not large and it is relevant only in the very early/very late universe. Hence, the purpose of such additional reconstruction is only to improve the cosmological predictions if the original theory does not pass correctly the precise observational cosmological bounds. For instance, in this way it is possible to formulate the modified gravity without finite-time future singularity.

The present reconstruction formulation shows that even if specific realistic modified gravity does not pass correctly some cosmological bounds (for instance, does not lead to correct cosmological perturbations structure) it may be improved with eventually desirable result. Hence, the successful development of such method adds very strong argument in favour of unified gravitational alternative for inflation, dark energy and dark matter.

Chapter 6

Viable F(R) cosmology and the presence of phantom fluids

[1]Our first aim in this chapter will be to build a reliable cosmological model by using, as starting point, modified gravity theories of the family of the so-called $F(R)$ theories which comprise the class of viable models, e.g., those having an $F(R)$ function such that the theory can pass all known local gravity tests (see [10, 88, 105, 109, 176, 219, 234, 235, 237, 248, 258, 286, 291]). In the next section, we will investigate cosmological evolution as coming from $F(R)$ gravity. We will consider, for $F(R)$, some candidates to produce inflation and cosmic acceleration in a unified fashion. In particular, we investigate in detail the behavior of $F(R)$ gravity as the contribution of a perfect fluid. As a crucial novelty, an additional matter fluid will be included which may play, as we shall see, quite an important cosmological role. In fact, it may decisively contribute to the two accelerated epochs of the Universe, what is to say that we will study a model where dark energy consists of two separate contributions. The possibilities to obtain precision cosmology are enhanced in this way.

Two main cases will be discussed: $F(R)$ cosmology with a constant equation of state (EoS) fluid, and $F(R)$ cosmology in the presence of a phantom fluid. In the last case, a couple of specific example will be worked through in detail, namely one with a phantom fluid with constant EoS and, as a second example, a fluid with dynamical EoS of the type proposed in Chapter 3.

As it is well known, $F(R)$ gravity can be written in terms of a scalar field—quintessence or phantom like—by redefining the function $F(R)$ with the use of a scalar field, and then performing a conformal transformation, what yields to the so-called Einstein frame. It has been shown that, in general, for any given $F(R)$ the corresponding scalar-tensor theory can, in principle, be obtained, although the solution is going to be very different from one case to another. Also attention will be paid to the reconstruction of $F(R)$ gravity from a given scalar-tensor theory. It is known [60] that the phantom case in scalar-tensor theory does not exist, in general, when starting from $F(R)$ gravity. In fact, the conformal transformation becomes complex when the phantom barrier is crossed, and therefore the resulting $F(R)$ function becomes complex. We will see that, to avoid this hindrance, a dark fluid can be used in order to produce the phantom behavior in such a way that the $F(R)$ function reconstructed from the scalar-tensor theory continues to be real. We will prove, in an explicit manner, that an $F(R)$ theory can indeed be constructed from a phantom model in a scalar-tensor theory, but where the scalar field does not behave as a phantom field (in which case the action for $F(R)$ would be complex). Moreover, we will explicitly show that very interesting

[1]This Chapter is based on: [138].

and quite simple $F(R)$ models crossing the phantom divide can be constructed.

6.1 Viable $F(R)$ gravities

Our first aim in this chapter will be to construct reliable cosmological models by using, as starting point, modified gravity theories of the family of the so-called $F(R)$ theories which comprise the class of viable models, e.g., those having an $F(R)$ function such that the theory can pass all known local gravity tests. We now consider the action corresponding to one of these theories which, aside from the gravity part, also contains a matter contribution, namely

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2\kappa^2}(R+F(R)) + L_m\right] \ . \tag{6.1}$$

Here $\kappa^2 = 8\pi G$, and L_m stands for the Lagrangian corresponding to matter of some kind. Note that the first term in (6.1) is just the usual Hilbert-Einstein action, and the $F(R)$ term can be considered, as it was shown in Chapter 4, as the dynamical part of some kind of perfect fluid, which may constitute an equivalent to the so-called dark fluids. The field equations corresponding to action (6.1) are obtained by variation of this action with respect to the metric tensor $g_{\mu\nu}$, what yields

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + R_{\mu\nu}F'(R) - \frac{1}{2}g_{\mu\nu}F(R) + g_{\mu\nu}\Box F'(R) - \nabla_\mu\nabla_\nu F'(R) = \kappa^2 T_{\mu\nu} \ . \tag{6.2}$$

Here the primes denote derivatives with respect to R. We assume a flat FLRW metric, then the Friedmann equations are obtained as the 00 and the ij components. They take the form

$$\begin{aligned} 3H^2 &= \kappa^2\rho_m - \frac{1}{2}F(R) + 3(H^2+\dot{H})F'(R) - 18F''(R)(H^2\dot{H} + H\ddot{H}) \ , \\ -3H^2 - 2\dot{H} &= \kappa^2 p_m + \frac{1}{2}F(R) - (3H^2+\dot{H})F'(R) - \Box F'(R) - HF''(R)\dot{R} \ , \end{aligned} \tag{6.3}$$

where $H(t) = \dot{a}/a$. Note that both equations are written in such a way that the $F(R)$-terms are put on the matter side; thus we may define an energy density and a pressure density for these $F(R)$-terms, as follows

$$\begin{aligned} \rho_{F(R)} &= -\frac{1}{2}F(R) + 3(H^2+\dot{H})F'(R) - 18F''(R)(H^2\dot{H} + H\ddot{H}) \ , \\ p_{F(R)} &= \frac{1}{2}F(R) - (3H^2+\dot{H})F'(R) - \Box F'(R) - HF''(R)\dot{R} \ . \end{aligned} \tag{6.4}$$

Then, the Friedmann equations (6.3) take a simple form, with two fluids contributing to the scale factor dynamics. From the energy and pressure densities defined in (6.4), one can obtain the EoS for the dark fluid, defined in terms of the $F(R)$ components. This is written as

$$\begin{aligned} w_{F(R)} &= \frac{\frac{1}{2}F(R) - (3H^2+\dot{H})F'(R) - \Box F'(R) - HF''(R)\dot{R}}{-\frac{1}{2}F(R) + 3(H^2+\dot{H})F'(R) + \Box F'(R) - \nabla_0\nabla^0 F'(R)} \\ \longrightarrow \qquad p_{F(R)} &= -\rho_{F(R)} + 2\dot{H}F'(R) - \nabla_0\nabla^0 F'(R) - HF''(R)\dot{R} \ . \end{aligned} \tag{6.5}$$

The EoS, Eq. (6.5), defines a fluid that depends on the Hubble parameters and its derivatives, so it may be considered as a fluid with inhomogeneous EoS (see [81]). In absence of any kind of matter, the dynamics of the Universe are carried out by the $F(R)$-component, which may be chosen so that it reproduces (or at least contributes) to the early inflation and late-time acceleration epochs. In order to avoid serious problems with known physics, one has to choose the $F(R)$ function in order that the theory contains flat solutions and passes also the local gravity tests. To reproduce the whole history of the Universe, the following conditions on the $F(R)$ function have been proposed (see [234, 237, 235]):

i) Inflation occurs under one of the following conditions:

$$\lim_{R\to\infty} F(R) = -\Lambda_i \tag{6.6}$$

or

$$\lim_{R\to\infty} F(R) = \alpha R^n \ . \tag{6.7}$$

In the first situation (6.6), the $F(R)$ function behaves as an effective cosmological constant at early times, while the second condition yields accelerated expansion where the scale factor behaves as $a(t) \sim t^{2n/3}$.

ii) In order to reproduce late-time acceleration, we can impose on function $F(R)$ a condition similar to the one above. In this case the Ricci scalar has a finite value, which is assumed to be the current one, so that the condition is expressed as

$$F(R_0) = -2R_0 \quad F'(R_0) \sim 0 \ . \tag{6.8}$$

Hence, under these circumstances, the $F(R)$ term is able to reproduce the two different accelerated epochs of the universe history. An interesting example that satisfies these conditions has been proposed in Ref. [176]:

$$F(R) = \frac{\mu^2}{2\kappa^2}\frac{c_1(\frac{R}{c_1})^k + c_3}{c_2(\frac{R}{\mu^2})^k + 1} \ . \tag{6.9}$$

This model is studied in detail in Ref. [109], where it is proven that the corresponding Universe solution goes, in its evolution, through two different De Sitter points, being one of them stable and the other unstable and which can be identified as corresponding to the current accelerated era and to the inflationary epoch, respectively. Thus, the model (6.9) is able to reproduce both accelerated epochs when the free parameters are conveniently chosen—as was done in Ref. [109]—in such way that the the F(R) theory in (6.9) produces two de Sitter epochs and grateful exit from the inflationary stage is achieved. In the same way, from a similar example given in Ref. [237], we can consider the function

$$F(R) = \frac{R^n(\alpha R^n - \beta)}{1 + \gamma R^n}. \tag{6.10}$$

This function, represented in Fig. 6.1, gives rise to inflation at the early stages of the Universe, assuming the condition (6.7) holds, while for the current epoch it behaves as an effective cosmological constant. This is explicitly seen in Fig. 6.1, where function (6.10) is represented for the specific power $n = 2$.

Function (6.10) leads, at the current epoch, to a perfect-fluid behavior with an EoS given by $p_{F(R)} \sim -\rho_{F(R)}$. In the next section such kind of $F(R)$ functions will be considered, with the inclusion of a matter Lagrangian in the action (6.1), and the corresponding cosmological evolution will be studied. We will see that both inflation and the current acceleration can indeed be produced by the $F(R)$ fluid, provided other components are allowed to contribute too.

6.2 Cosmological evolution from viable $F(R)$ gravity with a fluid

In this section we will discuss cosmological evolution as coming from $F(R)$-gravity. We consider the function $F(R)$ as given by (6.10). Functions of this kind have been shown to yield viable models which comply with all known local gravity tests (see, e.g., [235]), and they are good candidates to produce inflation and cosmic acceleration in a unified fashion. We will study in detail the behavior of $F(R)$-gravity, by

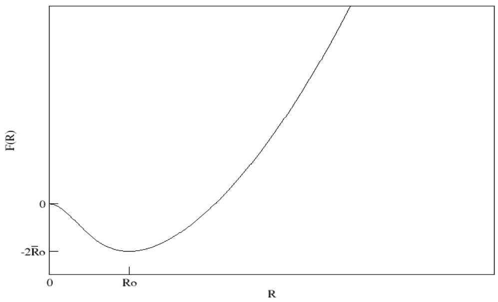

Figure 6.1: The function $F(R)$ as given by (6.10) for $n=2$. We see that at the current epoch ($R \sim R_0$), $F(R)$ behaves as a cosmological constant while for $R \to \infty$ (inflation) its values grow as a power law.

considering it as the contribution of a perfect fluid in the way already explained in the preceding section. As a crucial novelty, an additional matter fluid will be here incorporated, which may play, as we shall see, an important cosmological role. It may decisively contribute to the two accelerated epochs of the Universe, what is to say that we study a model where dark energy consists of two separate contributions. Some examples of phantom evolution will be then discussed, in which the $F(R)$ contribution acts as a cosmological constant and the additional fluid behaves as a phantom field, what opens again interesting new venues.

6.2.1 $F(R)$ cosmology with a constant EoS fluid $p_m = w_m \rho_m$

We consider a Universe governed by action (6.1), where the $F(R)$ function is given by (6.10). The matter term is represented by a perfect fluid, whose equation of state is $p_m = w_m \rho_m$ (where $w_m =$ cons.) In this case, by considering this $F(R)$ term as a perfect fluid—with energy and pressure densities given by (6.4)—the Friedmann equations reduce to Eqs. (6.3). For simplicity, we will study the case where $n=2$; then, the function $F(R)$ of our specific model reads (the study can be extended to other values of n without much problem)

$$F(R) = \frac{R^2(\alpha R^2 - \beta)}{1+\gamma R^2} \, . \tag{6.11}$$

First of all, let us explain qualitatively what the aim of this model is. For simplicity, we neglect the contribution of matter, so the first Friedmann equation yields

$$\begin{aligned} 3H^2 = & -\frac{R^2(\alpha R^2-\beta)}{2(1+\gamma R^2)} + 3(H^2+\dot{H})\frac{2R(\alpha\gamma R^4 - 2\alpha R^2 - \beta)}{(1+\gamma R^2)^2} - \\ & - 18F''(R)(H^2\dot{H} + H\ddot{H})\frac{2\alpha\gamma^2 R^6 + 20\alpha\gamma R^4 + 6(\beta\gamma-\alpha)R^2 - 2\beta}{(1+\gamma R^2)^3} \, , \end{aligned} \tag{6.12}$$

where $R = 6(2H^2+\dot{H})$. It can be rewritten as a dynamical system (see Ref. [109]):

$$\dot{H} = C \, , \quad \dot{C} = F_1(H, C) \, , \tag{6.13}$$

and it can be shown that its critical points are those which give a constant Hubble rate ($\dot{H}=0$), i.e., the points that yield a de Sitter solution of the Friedmann equations. We can now investigate the existence of

these points for the model (6.11) by introducing the critical points $H = H_0$ in the Friedmann equation (6.12)

$$3H_0^2 = -\frac{R^2(\alpha R^2 - \beta)}{2(1+\gamma R^2)} + 3H_0^2 \frac{2R(\alpha\gamma R^4 - 2\alpha R^2 - \beta)}{(1+\gamma R^2)^2} \ . \tag{6.14}$$

To simplify, it can be rewritten in terms of the Ricci scalar $R_0 = 12H_0^2$, what yields

$$\frac{\gamma}{4}R_0^5 - \gamma\beta R_0^4 + \frac{\gamma}{2}R_0^3 + \frac{1}{4}R_0 = 0 \ . \tag{6.15}$$

This can be solved, so that the viable de Sitter points (positive roots) are found explicitly. A very simple study of Eq. (6.15), by using Descartes' rule of signs, leads to the conclusion that Eq. (6.15) can have either two or no positive roots. As shown below, one of these roots is identified as an effective cosmological constant that produces the current accelerated expansion of the Universe, while the second root can produce the inflationary epoch under some initial conditions. Then, the model described by the function (6.11) is able to unify the expansion history of the Universe. In order to get a grateful exit from the inflationary epoch, stability in the vicinity of the critical points needs to be studied: the corresponding de Sitter point during inflation must be unstable. This can be achieved, as is already known for the function (6.9), by choosing specific values of the free parameters. Even in the case of stable dS inflation, exit from it can be achieved by coupling it with matter, by the effect of a small non-local term (or by some other mechanism).

Let us now study the details of this same model at the current epoch, when it is assumed that F(R) produces the cosmic acceleration and where the matter component is taken into account. Function (6.11) is depicted in Fig. 6.1; its minimum is attained at $R = R_0$, which is assumed to be the current value of the Ricci scalar. Further, $F(R)$ as given by (6.11) behaves as a cosmological constant at present time. Imposing the condition $\beta\gamma/\alpha >> 1$ on the otherwise free parameters, the values of R_0 and $F(R_0)$ are then given by [237]

$$R_0 \sim \left(\frac{\beta}{\alpha\gamma}\right)^{1/4} , \qquad F'(R_0) = 0 \ , \qquad F(R_0) = -2\tilde{R}_0 \sim -\frac{\beta}{\gamma} \ . \tag{6.16}$$

For simplicity, we shall study the cosmological evolution around the present value of the Ricci scalar, $R = R_0$, where (6.11) can be expressed as $F(R) = -2\tilde{R}_0 + f_0(R-R_0)^2 + O\left((R-R_0)^3\right)$, the solution for the Friedmann equations (6.3) can be written as $H(t) = H_0(t) + \delta H$, where at zero order the solution is the same as in the case of a cosmological constant, namely

$$H(t) = \sqrt{\frac{\tilde{R}_0}{3}} \coth\left(\frac{(1+w_m)\sqrt{3\tilde{R}_0}}{2}t\right) \ . \tag{6.17}$$

As pointed out in Ref. [237], the perturbations δH around the current point $R = R_0$ may be neglected. Therefore, we can study the evolution of the energy density (6.4) for the $F(R)$ term as the EoS parameter defined by (6.5) around the minimum of the $F(R)$ function, which is assumed to be the present value of the Ricci scalar. For such purposes, it is useful to rewrite the Hubble function (6.17) as a function of the redshift z , instead of t . The relation between both variables can be expressed as

$$\frac{1}{1+z} = \frac{a}{a_0} = \left[A\sinh\left(\frac{(1+w_m)\sqrt{3\tilde{R}_0}}{2}\,t\right)\right]^{\frac{2}{3(1+w_m)}} \ , \tag{6.18}$$

where a_0 is taken as the current value of the scale parameter, and $A^2 = \rho_{0m}a_0^{-3(1+w_m)}$, being ρ_{0m} the current value of the energy density of the matter contribution. Hence, the Hubble parameter (6.17) is

expressed as a function of the redshift z, as

$$H^2(z) = \frac{\tilde{R}_0}{3}\left[A^2(1+z)^{3(w_m+1)} + 1\right] . \tag{6.19}$$

Thus, the model characterized by the action (6.1) with the function (6.11) and a matter fluid with constant EoS, depends on the free parameters (α, β, γ) contained in the expression of $F(R)$ and also on the value of the EoS parameter (w_m). When imposing the minimum value for the function $F(R)$ to take place at present time ($z = 0$), the free parameters can be adjusted with the observable data. Then we study the behavior of our model close to $z = 0$, where the contributions of non-linear terms produced by the function (6.11) are assumed not to modify the solution (6.19). In spite of the Hubble parameter being unmodified, for z close to zero, the energy density $\rho_{F(R)}$ will in no way remain constant for small values of the redshift. To study the behavior of the $F(R)$ energy density given by (6.4) it is convenient to express it as the cosmological parameter, $\Omega_{F(R)} = \frac{\rho_{F(R)}}{3H^2(z)}$, which can be written as

$$\Omega_{F(R)}(z) = -\frac{F(R)}{6H^2(z)} + \left[1 + \frac{\dot{H}(z)}{H^2(z)}\right] F'(R) - 18F''(R)\left[\dot{H}(z) + \frac{\ddot{H}(z)}{H(z)}\right] , \tag{6.20}$$

where the Ricci scalar is given by $R = 6[2H^2(z) + \dot{H}(z)]$. This expression (6.20) can be studied as a function of the redshift. As we are considering the solution (6.19), which has been calculated near $z = 0$, where the $F(R)$ function has a minimum, the second term in the expression (6.20) is negligible as compared with the other two terms ($F''(R_0)R_0/F'(R_0) \sim \sqrt{\beta\gamma/\alpha} >> 1$). This aproximation, as the solution (6.19), is valid for values of the Ricci scalar close to R_0, where the higher derivatives of $F(R)$ are small compared with the function. The aproximation is no longer valid when the F(R) derivatives are comparable with F(R). Then, as discussed above, the free parameters of the model can be fitted by the current observational values of the cosmological parameters, and by fixing the minimum of $F(R)$ to occur for $z \sim 0$.

We shall use the value for the Hubble parameter $H_0 = 100\,h$ km s^{-1} Mpc^{-1} with $h = 0.71 \pm 0.03$ and the matter density $\Omega_m^0 = 0.27 \pm 0.04$ given in Ref. [283]. In Fig. 6.2, the evolution of the $F(R)$ energy density (6.20) is represented for the model described by (6.11), where the matter content is taken to be pressureless ($w_m = 0$). The cosmological evolution shown corresponds to redshifts from $z = 1.8$ to $z = 0$. For redshifts larger than $z = 1.8$ perturbations around the solution (6.17) are non-negligible, and the expression for the Hubble parameter (6.17) is no more valid. However, in spite of the fact that the evolution shown in Fig. 6.2 is not a complete picture of the $F(R)$ model given by (6.11), it still provides an illustrative example to compare it to the standard ΛCDM model, which is also represented in Fig. 6.2, around the present time. As shown in Fig. 6.2, both models have two common points for $z = 0$ and $z = 1.74$, while the behavior of each model between such points is completely different from one another. This result shows the possible differences between $F(R)$ models of the type (6.10) and the ΛCDM model, although probably other viable models for $F(R)$ gravity may give a different adjustment to the ΛCDM model. Furthermore, the effective EoS parameter for the $F(R)$ fluid, $w_{F(R)}$, given by the expression (6.5), is plotted in Fig. 6.3, again as a function of redshift. It is shown there that $w_{F(R)}$ is close to -1 for $z = 0$, where the $F(R)$ fluid behaves like an effective cosmological constant, while it grows for redshifts up to $z = 1.5$, where it reaches a zero value. According to the analysis of observational dataset from Supernovae (see Ref.[195]), the results obtained for the evolution of the EoS parameter, represented in Fig. 6.3, are allowed by the observations.

As a consequence, the $F(R)$ model given by (6.11), and where the $F(R)$ fluid behaves as an effective cosmological constant, is able indeed to reproduce the same behavior at present time as the ΛCDM model. On the other hand, as was pointed out in the section above, the $F(R)$ model given by (6.11) also reproduces the accelerated expansion of the inflation epoch, so that the next natural step to undertake

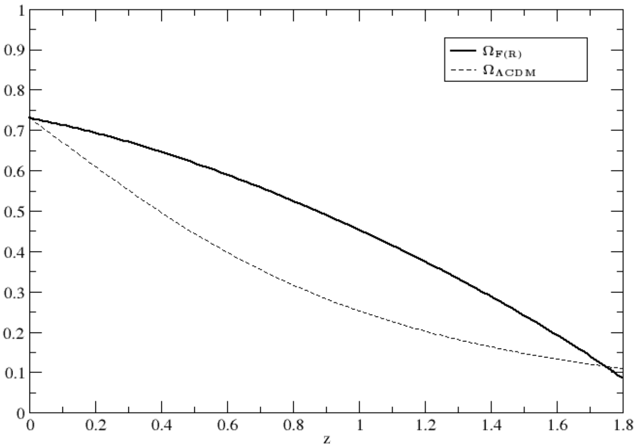

Figure 6.2: Evolution of the cosmological parameter from dark energy versus redshift, such for $F(R)$ theory as for ΛCDM model.

with this kind of models should be to study their complete cosmological histories, and the explicit details allowing for a grateful exit from inflation, what should demonstrate their true potential. Also note that, in order to obtain a realistic well behaved model, further analysis should be carried out, as the comparison with the luminosity distance from Supernovae, or the data from CMB surveys, although this is a major task, that will be left for future work. Furthermore, $F(R)$ functions of this kind might even lead to a solution of the cosmological constant problem, by involving a relaxation mechanism of the cosmological constant, as was proposed in Ref. [296]. The effective cosmological constant obtained could eventually adjust to the observable value. This will also require a deeper investigation.

6.2.2 $F(R)$ cosmology in presence of a phantom fluid

According to several analysis of observational data (see [256, 179]), the effective EoS parameter of the physical theory that governs our Universe should quite probably be less than -1, what means that we should be ready to cope with phantom behavior. This possibility has been explored in $F(R)$ theory [60], where the possibility to construct an $F(R)$ function that reproduces this kind of behavior has been demonstrated, and where the possible future singularities envisaged, as the Big Rip, so common in phantom models, do take place. In Chapter 4, an example with phantom behavior that ends in a Big Rip singularity was shown, as well as another phantom model that avoids future singularities. In this last example, the universe has a phantom behavior at large time but there is no Big Rip singularity. However, the main problem of those models is that the corresponding $F(R)$ function is not well constrained at small scales.

We will now study the behavior of the cosmological evolution when a phantom fluid is introduced that contributes to the accelerated expansion. A phantom fluid can be described, in an effective way, by an EoS $p_{ph} = w_{ph}\rho_{ph}$, where $w_{ph} < -1$. This EoS can be achieved, for example, with a negative kinetic term or in the context of scalar-tensor theory. The microphysical study of this kind of fluids is a problem of fundamental physics that has been studied in several works (see for example [70, 96]), as it is also the

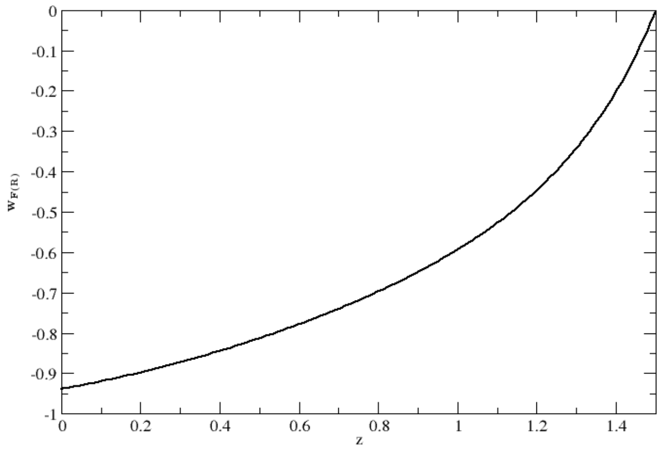

Figure 6.3: Effective EoS parameter $w_{F(R)}$ versus redshift. It takes values close to -1 for $z=0$, and it grows for higher redshifts.

study of stability of the solutions. But it is not the aim for this chapter to discuss, in this depth, the features of phantom fields, and it will suffice for our purpose to consider the effective EoS that describes a phantom fluid. The $F(R)$ function we will consider is the same as above, given by (6.10), which will contribute as an effective cosmological constant at present time. The motivation for this case study comes, as will be further explored, from the difficulty one encounters in constructing *viable* $F(R)$ functions which produce phantom epochs, and this problem has a translation in the scalar-tensor picture in the Einstein frame too. As recent observational data suggest (see [179, 256]), the effective EoS parameter for dark energy is around -1, so that the phantom case is allowed (and even still favored) by recent, accurate, and extensive observations.

Example 1. First, we consider a phantom fluid with constant EoS, e.g., $p_{ph} = w_{ph}\rho_{ph}$, where $w_{ph} < -1$ is a constant. As in the case above, the $F(R)$ function will be given by (6.11), and it will be assumed that its current value is attained at the minimum $R_0 = (\beta/\alpha\gamma)^{1/4}$. We here neglect other contributions, as the dust term studied previously. Friedmann equations take the form

$$H^2 = \frac{\kappa^2}{2}\rho_{ph} + \frac{\tilde{R}_0}{3} \ , \quad \dot{H} = -\frac{\kappa^2}{3}\rho_{ph}(1+w_{ph}) \ . \tag{6.21}$$

An expanding solution (note that a contracting solution can be obtained from the above equations too) for these equations is given by

$$H(t) = \frac{3}{2|1+w_{ph}|(t_s - t)} + \sqrt{\frac{\tilde{R}_0}{3}} \ , \tag{6.22}$$

where t_s is called the Rip time, that means the instant where a future Big Rip singularity will take place. Although for t much bigger than the present time, the solution (6.22) is not valid anymore — because perturbations, due to the derivatives of the function $F(R)$, become large— and as the Ricci scalar $R = 6(2H^2 + \dot{H})$ grows with time, this model will behave as $F(R) \sim R^2$ for large times, which is known

to produce accelerated expansion, and whose behavior is described by

$$H(t) \propto \frac{h_0}{t_s - t} , \tag{6.23}$$

where $h_0 = \frac{4}{3|w_{ph}+1|}$. And the effective EoS parameter for large time is

$$w_{eff} = -1 - \frac{4}{3h_0} . \tag{6.24}$$

That is to say, the universe will go through a super-accelerated expansion stage due to the phantom fluid and to the contribution coming from $F(R)$, until it reaches the Big Rip singularity. In this case where no other contribution, as dust matter or radiation, is taken into account, late-time acceleration comes from the phantom behavior when the dark fluid component dominates, while the Universe behaves as dark energy when the $F(R)$ term is the dominant one (as was in absence of this phantom fluid). In other words, the universe would not accelerate as a phantom one alone, but it will as a dark energy fluid with $w_{eff} \geq -1$. This is a fundamental point: the additional dark fluid is essential for a universe described by the $F(R)$ function given in (6.11) to display a phantom transition.

We can study the evolution of this phantom fluid by using the continuity equation:

$$\dot{\rho}_{ph} + 3H\rho_{ph}(1 + w_{ph}) = 0 , \tag{6.25}$$

which can be solved, at the present time, when the Hubble parameter is expressed by Eq. (6.22). Then, the following solution is obtained

$$\rho_{ph} = \rho_{0ph} \frac{\mathrm{e}^{|1+w_{ph}|\sqrt{3\tilde{R}_0}t}}{(t_s - t)^{9/2}} , \tag{6.26}$$

where ρ_{0ph} is an integration constant. As we can see, the energy density for the phantom fluid grows with time until the Rip value is reached where the energy density becomes infinite. On the other hand, the evolution of the $F(R)$ term may be studied qualitatively by observing the expression given for the energy density of this fluid in (6.4), so that this evolution is similar to the one in the above case. That is, as the value of the Ricci scalar increases with time, it is natural to suppose that, in the past, the energy density belonging to $F(R)$ had smaller values than at present, so that the matter dominated epoch could occur when the $F(R)$ fluid and the phantom fluid were much less important than they are now. Again, for this kind of model the $F(R)$ contribution amounts currently to an effective cosmological constant which drives the universe's acceleration.

Example 2. As a second example of a phantom fluid, we consider one with a dynamical EoS of the type proposed in Chapter 3. Here, a dark fluid is present which has an inhomogeneous EoS that may depend on the proper evolution of the Universe. This kind of EoS can be derived from the dynamics of an scalar field with some characteristic potential and a variable kinetic term, or either it may be seen as the effective EoS corresponding to the addition of various components that fill up our Universe. An EoS of this kind can be written as

$$p_{ph} = w\rho_{ph} + g(H, \dot{H}, \ddot{H} ...; t) , \tag{6.27}$$

where w is a constant and g an arbitrary function. The interesting point is that it is possible to specify a function g so that a complete solution of the Friedmann equations is obtained. In this case, our aim is to study a fluid that at present (or in the near future) can behave as phantom; to that purpose we choose $w = -1$, and the role of the g function will be to determine when exactly the phantom barrier is crossed.

Thus, our model will be described by the phantom fluid in (6.27) and the function $F(R)$ of (6.11), while the matter component can be neglected. As an example, the EoS for the dark fluid is given by

$$p_{ph} = -\rho_{ph} - \left(\frac{4}{\kappa^2} h'(t) + p_{F(R)} + \rho_{F(R)} \right) , \tag{6.28}$$

where the prime denotes derivative with respect to time, and $p_{F(R)}$ and $\rho_{F(R)}$ are the energy densities defined in (6.4). As it is shown, through the EoS defined in (6.28), the dark fluid will contribute to the acceleration of the Universe as a dark energy, and subsequently as a phantom fluid when it crosses the barrier $w_{ph} < -1$. We thus see that in this model accelerated expansion comes from two contributions, the $F(R)$ term (6.11) and the dark fluid one (6.28), so that the accelerated expansion stage could cross the phantom barrier if the dark fluid dominates and contributes as a phantom one. The difference with respect to the former model is that in the present case the dark fluid changes its behavior in the course of the expansion history (this will be seen again in an example below). As the EoS given in (6.28) can be rewritten in terms of the Ricci scalar R, it can be seen as additional terms to our F(R) function, in order to get the transition to the phantom epoch in the context of F(R) gravity. With the EoS (6.28), the Friedmann equations can be solved, the following solution being found

$$H(t) = h(t) \ . \tag{6.29}$$

Different solutions can be constructed by specifying the function $h(t)$. We are here interested in those solutions that give rise to a phantom epoch; for those cases the following splitting of the Hubble parameter is relevant

$$H(t) = \frac{H_0}{t} + \frac{H_1}{t_s - t} , \tag{6.30}$$

where H_0, H_1 and t_s (Rip time) are positive constants. This function describes a Universe that starts in a singularity, at $t = 0$—which may be identified with the Big Bang one—then evolves to a matter dominated epoch, after which an accelerated epoch starts which is dominated by an effective cosmological constant and, finally, the Universe enters into a phantom epoch that will end in a Big Rip singularity. As in the cases above, the condition (6.16) remains, that is $F'(R_0) = 0$, where R_0 is the present value of the Ricci scalar. Fig. 6.4 is an illustration of how this model works: we see there that the Universe goes through a decelerated epoch until it enters a region where the evolution of its expansion has constant Hubble parameter and the $F(R)$ term behaves as an effective cosmological constant. Finally, at t_{ph} the Universe enters into a super-accelerated phase which is dominated by the dark fluid of Eq. (6.28), until it eventually reaches the Big Rip singularity at $t = t_s$. The aim of this model is that the crossing phantom barrier takes place very softly, as it is seen in Fig. 6.4 due to the two contributions to the acceleration of the Universe expansion. Alternatively, the evolution of the EoS parameter for the dark fluid (6.28) can be written as

$$w_{ph} = -1 - \frac{\frac{4}{\kappa^2} h'(t) + p_{F(R)} + \rho_{F(R)}}{\frac{6}{\kappa^2} h^2(t) - \rho_{F(R)}} , \tag{6.31}$$

and we may look at its asymptotic behavior,

$$\begin{aligned} \text{For} \quad 0 < t << t_0 \longrightarrow w_{ph} \sim -1 + \frac{2(H_0+4)}{3(3-2H_0^2)} \ . \\ \text{For} \quad t \sim t_0 \longrightarrow w_{ph} \sim -1 - \frac{4}{\kappa^2} \dot{H} \ . \\ \text{For} t_0, t_{ph} << t < t_s \longrightarrow w_{ph} \sim -1 - \frac{2}{9\left(\frac{37H_1^2 - 4H_1 + 18}{H_1}\right)} \ . \end{aligned} \tag{6.32}$$

We thus see that, at the starting stages of the Universe, the dark fluid contributes to the deceleration of its expansion. For t close to the present time, t_0, it works as a contribution to an effective cosmological

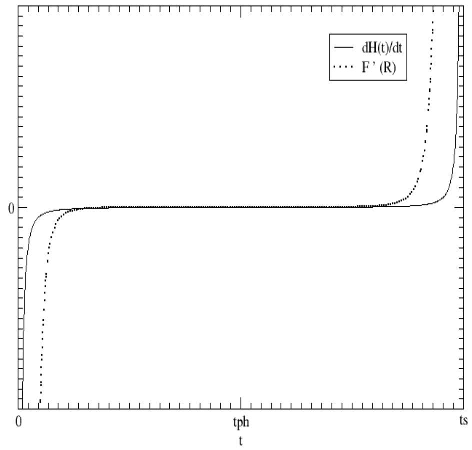

Figure 6.4: The first derivative of the Hubble parameter is shown here to illustrate the different epochs of the Universe evolution (full curve). Also, the first derivative of the $F(R)$ term with respect to R is represented as a function of time (dotted curve); the constant Hubble parameter region is given by a constant $F(R)$, which plays the role of an effective cosmological constant. Starting at $t = t_{ph}$, the Universe enters into a phantom region dominated by the dark fluid, which ends at $t = t_s$ when the Big Rip singularity takes over.

constant and, after $t = t_{ph}$, it gives rise to the transition to a phantom era of the cosmos, which could actually be taking place nowadays at some regions of it. Our hope is that it could even be observable with specific measurements. Finally, for t close to the Rip time, the Universe becomes completely dominated by the dark fluid, whose EoS is phantom at that time. This model, which is able to accurately reproduce the dark energy period, may still be modified in such a way that the epoch which is dominated by the effective cosmological constant, produced by the $F(R)$ term and by the dark fluid contribution, becomes significantly shorter. This is supposed to happen when a matter term is included.

To finish this section, the inclusion of a dark fluid that behaves as a phantom one gives rise to a super-accelerated phase, as compared with the case where just the viable $F(R)$ term contributes. In the two examples here studied, we have proven that, while the $F(R)$ term contributes as an effective cosmological constant, the dark fluid contribution produces the crossing of the phantom barrier, and it continues to dominate until the end of the Universe in a Big Rip singularity. In other words, both the contribution of $F(R)$ and of the phantom fluid are needed. This is very clearly seen before, and the nice thing is that, because of this interplay, we have shown the appearance of nice properties of our model that no purely phantom model could have. Specifically, for our models above, $F(R)$ gravity together with phantom matter, the effective w value becomes in fact bigger than -1, so that we are able to show that $F(R)$ gravity can solve the phantom problem simply by making the phantom field to appear as a normal one.

6.3 Scalar-tensor theories and $F(R)$ gravity with a fluid

We now turn to the study of the solutions given above in the alternative, and more commonly used, scalar-tensor picture. In Refs. [1, 19, 60, 81, 232], it was pointed out that $F(R)$ gravity can be written in terms of a scalar field—quintessence or phantom like—by redefining the function $F(R)$ with the use of a convenient scalar field and then performing a conformal transformation. The scalar-tensor theory thus obtained provides a solution which is characterized by this conformal transformation, whose expression depends on the precise form of the $F(R)$ function. It has been shown that, in general, for any given $F(R)$, the corresponding scalar-tensor theory can in principle be obtained, although the solution is going to be very different from case to case. Also attention has been paid to the reconstruction of $F(R)$ gravity from a given scalar-tensor theory. It is also known (see [60]) that the phantom case in scalar-tensor theory does not allow, in general, the corresponding picture in $F(R)$ gravity. In fact, the conformal transformation becomes complex when the phantom barrier is crossed, and then the resulting $F(R)$ function becomes complex too. To avoid this hindrance, a dark fluid can be used, as in the models of the preceding section, in order to produce the phantom behavior in such a way that the reconstructed $F(R)$ function continues to be *real*. This point is important, and will be clearly shown below. Also to be remarked is the fact that scalar-tensor theories, commonly used in cosmology to reproduce dark energy (see Chapter 2 and 3), provide cosmological solutions whose stability should be studied in order to demonstrate the validity of the solution found. This has been investigated, e.g., in Ref. [70] and requires a very deep and careful analysis. For the purpose of the current chapter, we will here concentrate in the proof the existence of the solution in the scalar-tensor counterpart, and the corresponding stability study will be left for future work.

We start with the construction of the scalar-tensor theory from $F(R)$ gravity. The action (6.1) can be written as

$$S = \int d^4x\sqrt{-g}\left[P(\phi)R + Q(\phi) + L_m\right] \ , \tag{6.33}$$

which is known as the Jordan frame. Here $F(R)$ has been written in terms of a scalar field. To recover the action (6.1) in terms of $F(R)$, the scalar field equation resulting from the variation of the action (6.33) with respect to ϕ is used, which can be expressed as follows

$$P'(\phi)R + Q'(\phi) = 0 \ , \tag{6.34}$$

where the primes denote derivatives with respect to ϕ. Then, by solving equation (6.34) we get the relation between the scalar field ϕ and the Ricci scalar, $\phi = \phi(R)$. In this way, the original $F(R)$ function and the action (6.1) are recovered:

$$R + F(R) = P(\phi(R))R + Q(\phi(R)) \ . \tag{6.35}$$

Finally, the scalar-tensor picture is obtained by performing a conformal transformation on the action (6.33). The relation between both frames is given by

$$g_{E\mu\nu} = \Omega^2 g_{\mu\nu} \ , \quad \text{where} \quad \Omega^2 = P(\phi) \ , \tag{6.36}$$

where the subscript $_E$ stands for Einstein frame. A quintessence-like action results in the Einstein frame

$$S_E = \int d^4x\sqrt{-g_E}\left[R_E - \frac{1}{2}\omega(\phi)d_\mu\phi d^\mu\phi - U(\phi) + \alpha(\phi)L_{mE}\right] \ ,$$

where

$$\omega(\phi) = \frac{12}{P(\phi)}\left(\frac{d\sqrt{P(\phi)}}{d\phi}\right)^2 \ , \quad U(\phi) = \frac{Q(\phi)}{P^2(\phi)} \quad \text{and} \quad \alpha(\phi) = P(\phi) \ , \tag{6.37}$$

are the kinetic term, the scalar potential and the coupling function, respectively. Hence, by following the steps enumerated above, we can reconstruct the scalar-tensor theory described by the action (6.37) for a given $F(R)$ gravity. By redefining the scalar field $\phi = R$, and after combining Eqs. (6.34) and (6.35), the form of the two functions $P(\phi)$ and $Q(\phi)$ are found

$$P(\phi) = 1 + F'(\phi) \ , \quad Q(\phi) = F'(\phi)\phi - F(\phi) \ . \tag{6.38}$$

Hence, for a given solution in the Jordan frame (6.33), the solution in the corresponding quintessence/phantom scalar field scenario—i.e., in the Einstein frame (6.37)—is obtained by the conformal transformation (6.36), and it is given by

$$a_E(t_E) = \left[1 + F'(\phi(t))\right]^{1/2} a(t) \quad \text{where} \quad dt_E = \left[1 + F'(\phi(t))\right]^{1/2} dt \ . \tag{6.39}$$

We will be here interested in the phantom case. With that purpose, we analyze the model described in the above section by the $F(R)$ function (6.11) and the dark fluid with EoS (6.28), and whose solution is (6.30). For simplicity, we restrict the reconstruction to the phantom epoch, when the solution can be written as $H(t) \sim \frac{H_1}{t_s - t}$, and $F(R) \sim \frac{\alpha}{\gamma} R^2$. Using (6.38), the function $P(\phi(t))$ takes the form $P(t) \sim 2\frac{\alpha}{\gamma} R^2 = \frac{2\alpha H_1(H_1+1)}{\gamma} \frac{1}{(t_s - t)}$ as a function of time t in the Jordan frame. Then, using (6.39), the solution in the Einstein frame is found

$$t_E = -\sqrt{\frac{2\alpha H_1(H_1+1)}{\gamma}} \ln(t_s - t) \longrightarrow a_E(t_E) = \sqrt{\frac{2\alpha H_1(H_1+1)}{\gamma}} \exp\left[2\sqrt{\frac{\gamma}{2\alpha H_1(H_1+1)}} t_E\right] \ . \tag{6.40}$$

Through the relation between the time coordinates in both frames, we see that while for the Jordan frame there is a Big Rip singularity, at $t = t_s$, this corresponds in the Einstein frame to $t \to \infty$, so that the singularity is avoided, and there is no phantom epoch there. By analyzing the scale parameter in the Einstein frame, we realize that it describes a de Sitter Universe, while in the Jordan frame the Universe was described by a phantom expansion. As a consequence, we have shown that a phantom Universe in $F(R)$ gravity may be thoroughly reconstructed as a quintessence-like model, where the phantom behavior is lost completely. This has been achieved in a fairly simple example and constitutes an interesting result.

Let us now explore the opposite way. In this case, a phantom scalar-tensor theory is given and it is $F(R)$ gravity which is reconstructed. As was pointed out in Ref. [60], when a phantom scalar field is introduced, then the corresponding $F(R)$ function—which is reconstructed, close to the Big Rip singularity, by means of a conformal transformation that deletes the kinetic term for the scalar field—is in general complex. As a consequence, there is no correspondence in modified gravity when a phantom scalar produces a Big Rip singularity. However, in our case we will analyze a scalar-tensor theory which includes a phantom fluid that is responsible for the phantom epoch and for the Big Rip singularity. In this situation a real $F(R)$ gravity will be generically reconstructed, as we are going to see.

The action that describes the scalar-tensor theory is

$$S_E = \int d^4x \sqrt{-g_E} \left[R_E - \frac{1}{2}\omega(\phi) d_\mu \phi d^\mu \phi - U(\phi) + \alpha(\phi) L_{phE}\right] \ , \tag{6.41}$$

where $\alpha(\phi)$ is a coupling function and L_{phE} the Lagrangian for the phantom fluid in the Einstein frame. In our case, we consider a phantom fluid with constant EoS, $p_{phE} = w_{ph}\rho_{phE}$, with $w_{ph} < -1$. The Friedmann equations in this frame are written as

$$\begin{aligned} H_E^2 &= \frac{\kappa^2}{3}\left(\frac{1}{2}\omega(\phi)\dot{\phi}^2 + V(\phi) + \alpha(\phi)\rho_{phE}\right) \ , \\ \dot{H}_E &= -\frac{\kappa^2}{2}\Big(\omega(\phi)\dot{\phi}^2 + \alpha(\phi)\rho_{phE}(1 + w_{ph})\Big) \ . \end{aligned} \tag{6.42}$$

To solve the above equations, it turns out to be very useful to redefine the scalar field as $\phi = t_E$. Then, for a given solution $H_E(t_E)$, the kinetic term for the scalar field can be written as follows

$$\omega(\phi) = -\frac{\frac{4}{\kappa^2}(\dot{H}_E + 3(1+w_{ph})H_E^2) - (1+w_{ph})V(\phi)}{1-w_{ph}} \ . \tag{6.43}$$

To reconstruct $F(R)$ gravity, we perform a conformal transformation that deletes the kinetic term, namely

$$g_{\mu\nu E} = \Omega^2 g_{\mu\nu} \ , \quad \text{where} \quad \Omega^2 = \exp\left[\pm\sqrt{\frac{2}{3}}\kappa\int d\phi\sqrt{\omega(\phi)}\right] \ . \tag{6.44}$$

Note that for a phantom scalar field, that is defined by a negative kinetic term, the above conformal transformation would be complex, as remarked in Ref. [60] and as will be shown below. Thus, the reconstructed action would be complex, and no $F(R)$ gravity could be recovered. By means of the above conformal transformation, action (6.41) is given by

$$S = \int d^4x\sqrt{-g}\left[\frac{\mathrm{e}^{\left[\pm\sqrt{\frac{2}{3}}\kappa\int d\phi\sqrt{\omega(\phi)}\right]}}{2\kappa^2}R - \mathrm{e}^{\left[\pm 2\sqrt{\frac{2}{3}}\kappa\int d\phi\sqrt{\omega(\phi)}\right]}V(\phi) + L_{ph}\right] \ , \tag{6.45}$$

where L_{ph} is the lagrangian for the phantom fluid in the Jordan frame, whose energy-momentum tensor is related with the one in the Einstein frame by $T^{ph}_{\mu\nu} = \Omega^2 T^{phE}_{\mu\nu}$, and where we have chosen a coupling function $\alpha(\phi) = \Omega^{-4}$, for simplicity. By varying now the action (6.45) with respect to ϕ , the scalar field equation is obtained

$$R = \mathrm{e}^{\left[\pm\sqrt{\frac{2}{3}}\kappa\int d\phi\sqrt{\omega(\phi)}\right]}\left(4\kappa^2 V(\phi) \mp \sqrt{\frac{6}{\omega(\phi)}}V'(\phi)\right) \ , \tag{6.46}$$

which can be solved as $\phi = \phi(R)$, so that by rewriting action (6.45), the $F(R)$ gravity picture result:

$$F(R) = \frac{\mathrm{e}^{\left[\pm\sqrt{\frac{2}{3}}\kappa\int d\phi\sqrt{\omega(\phi)}\right]}}{2\kappa^2}R - \mathrm{e}^{\left[\pm 2\sqrt{\frac{2}{3}}\kappa\int d\phi\sqrt{\omega(\phi)}\right]}V(\phi) \ . \tag{6.47}$$

Hence, for some coupling quintessence theory described by action (6.41), it is indeed possible to obtain a *real* $F(R)$ theory, by studying the system in the Jordan frame through the conformal transformation (6.44).

To demonstrate this reconstruction explicitly, an example will now be given. As we are interested in the case of a phantom epoch close to the Big Rip singularity, we will start from a solution in the Einstein frame $H_E \sim \frac{1}{t_s - t_E}$, and a scalar potential given by $V(\phi) \sim (t_s - \phi)^n$, with $n > 0$. Then, the kinetic term (6.43) is written as

$$\omega(\phi) \sim \frac{-\frac{2}{\kappa^2}(2 + 3(1+w_{ph}))}{1-w_{ph}}\frac{1}{(t_s-\phi)^2} \ . \tag{6.48}$$

The solution in the Jordan frame is calculated by performing the conformal transformation (6.44)

$$(t_s - t_E) = \left[\left(1 \mp \frac{k}{2}\right)t\right]^{1/(1\mp k/2)} \quad \longrightarrow \quad a(t) \sim \left[\left(1 \mp \frac{k}{2}\right)t\right]^{-\frac{1\pm k}{1\mp k}} \ , \tag{6.49}$$

where $k = \sqrt{-\frac{8(1+3(1+w_ph))}{3(1-w_{ph})}}$ and $w_{ph} < -1$. Note that, in this case, we can construct two different solutions, and correspondingly two different $F(R)$ models, depending on the sign selected in Eq. (6.44). It is easy to see that the Big Rip singularity is thereby transformed, depending on the case, into an initial

singularity (+), or into an infinity singularity(-). By using (6.46) and (6.47), the following function $F(R)$ is recovered

$$F(R) \sim \frac{\left(\frac{R}{4\kappa^2}\right)^{\frac{\pm k}{n\pm k}}}{2\kappa^2} R - \left(\frac{R}{4\kappa^2}\right)^{\frac{n\pm 2k}{n\pm k}} . \tag{6.50}$$

To summarize, we have here shown, in an explicit manner, that an $F(R)$ theory can be actually constructed from a phantom model in a scalar-tensor theory where the scalar field does not behave as a phantom one (in that case the action for $F(R)$ would become complex). The above reconstruction procedure, where we have taken the $F(R)$ function of (6.11), can be generalized to other types of modified gravity models.

6.4 Discussion

We have seen in this chapter that the $F(R)$ model given by (6.11), and where the $F(R)$ fluid behaves as an effective cosmological constant, is able to reproduce the same behavior, at present time, as the Λ CDM model. On the other hand, this model gives rise to the accelerated expansion of the inflation epoch too, so that the natural next step to undertake with those models should be to study the complete cosmological history, in particular (what is very important) the explicit details for a grateful exit from inflation, what would demonstrate their actual potential. What is more, $F(R)$ functions of this kind might even lead to a solution of the cosmological constant problem, by involving a relaxation mechanism of the cosmological constant, as was indicated in [296, 26]). The effective cosmological constant thus obtained could eventually adjust to its precise observable value. This issue is central and deserves further investigation.

We have studied the behavior of the cosmological evolution when a phantom fluid is introduced that contributes to the accelerated expansion of the universe. The $F(R)$ function we have considered is the one given in (6.10), which contributes as an effective cosmological constant at present time. The motivation for this case study comes, as it will be further explored, from the difficulty one encounters to construct *viable* $F(R)$ functions which produce phantom epochs, and this has a representation in the scalar-tensor picture in the Einstein frame. As recent observational data suggest (see [256, 179]), the effective EoS parameter for dark energy is around -1, so that the phantom case is allowed—and actually favored by recent, extensive observations.

We thus have seen that, at the early stages of the universe history, the dark fluid contributes to the deceleration of its expansion. For t close to present time, t_0, it works as a contribution to an effective cosmological constant and, after $t = t_{ph}$, it gives rise to the transition to a phantom era of the universe, which could actually be taking place right now in some regions of it. Our hope is that it could be actually observable. Finally, for t close to the Rip time, the Universe becomes completely dominated by the dark fluid, whose EoS is phantom-like at that time. This model, which is able to reproduce the dark energy period quite precisely, may still be modified in such a way that the epoch dominated by the effective cosmological constant, produced by the $F(R)$ term and by the dark fluid contribution, becomes significantly shorter. This is the case when a matter term is included.

The inclusion of a dark fluid with phantom behavior gives rise to a super-accelerated phase, as compared with the case where just the viable $F(R)$ term contributes. In the two examples investigated in the chapter we have proven that, while the $F(R)$ term contributes as an effective cosmological constant, the dark fluid term produces the crossing of the phantom barrier, and it continues to dominate until the end of the universe in a Big Rip singularity. It is for this reason that the contribution of $F(R)$ and of the phantom fluid are both *fundamental*. This has been clearly explained in the chapter, and the nice thing is that, because of this interplay, we have shown the appearance of very nice properties of our model that no purely phantom model could have. This eliminates in fact some of the problems traditionally associated

with phantom models and makes this study specially interesting. In particular, for some of our models of $F(R)$ gravity together with phantom matter, the effective w value becomes in fact bigger than -1, so that we were able to show that $F(R)$ gravity can solve the phantom problem simply by making the phantom field to appear as a normal field.

To summarize, we have here shown, in an explicit manner, that an $F(R)$ theory can indeed be constructed from a phantom model in a scalar-tensor theory in which the scalar field does not behave as a phantom one (in the latter case the action for $F(R)$ would be complex). Moreover, very promising $F(R)$ models which cross the phantom divide can be constructed explicitly. The above reconstruction procedure, where we have taken (6.11) for the $F(R)$ function, can be generalized to other classes of modified gravity models.

Chapter 7

On ΛCDM model in modified $F(R, G)$ and Gauss-Bonnet gravitites

[1]In the preceedings chapters, it was studied and shown how modifications of the gravitational action with dependence on the Ricci scalar, can resolve some of the cosmological problems, specifically the acceleration phases of our Universe. in the present chapter, another kind of mofication of the Hilbert-Einstein action is presented, the so-called Gauss-Bonnet gravity, where the gravitational action includes functions of the Gauss-Bonnet invariant. This kind of theories have been investigated and may reproduce the cosmic history (see Refs. [5, 20, 21, 36, 77, 107, 108, 117, 118, 157, 168, 198, 226, 245, 241, 243, 244, 293, 312]). Here, we show that the Λ CDM model can be well explained with no need of a cosmological constant but with the inclusion of terms depending on the Gauss-Bonnet invariant in the action. Even more, it is shown that the extra terms in the action coming from the modification of gravity could behave relaxing the vacuum energy density, represented by a cosmological constant, and may resolve the so-called cosmological constant problem. To study and reconstruct the theory that reproduces such a model as well as other kind of solutions studied, we shall use the method proposed in Chapter 5, where the FLRW equations are written as functions of the so-called number of e-foldings instead of the cosmic time. The possible phantom epoch produced by this kind of theories is also explored, as well as other interesting cosmological solutions, where the inclusion of other contributions as perfect fluids with inhomogeneous EoS are studied.

7.1 Modified $R + f(G)$ gravity

We consider the following action, which describes General Relativity plus a function of the Gauss-Bonnet term (see Refs. [226, 241]):

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2\kappa^2}R + f(G) + L_m\right] , \tag{7.1}$$

where $\kappa^2 = 8\pi G_N$, G_N being the Newton constant, and the Gauss-Bonnet invariant is defined as usual:

$$G = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma} \ . \tag{7.2}$$

[1]This Chapter is based on: [132, 213]

By varying the action over $g_{\mu\nu}$, the following field equations are obtained:

$$0=\frac{1}{2k^2}(-R^{\mu\nu}+\frac{1}{2}g^{\mu\nu}R)+T^{\mu\nu}+\frac{1}{2}g^{\mu\nu}f(G)-2f_GRR^{\mu\nu}+4f_GR^{\mu}_{\rho}R^{\nu\rho}$$

$$-2f_GR^{\mu\rho\sigma\tau}R^{\nu}_{\rho\sigma\tau}-4f_GR^{\mu\rho\sigma\nu}R_{\rho\sigma}+2(\nabla^{\mu}\nabla^{\nu}f_G)R-2g^{\mu\nu}(\nabla^2f_G)R-4(\nabla_{\rho}\nabla^{\mu}f_G)R^{\nu\rho}$$

$$-4(\nabla_{\rho}\nabla^{\nu}f_G)R^{\mu\rho}+4(\nabla^2f_G)R^{\mu\nu}+4g^{\mu\nu}(\nabla_{\rho}\nabla_{\sigma}f_G)R^{\rho\sigma}-4(\nabla_{\rho}\nabla_{\sigma}f_G)R^{\mu\rho\nu\sigma}\,, \tag{7.3}$$

where we made the notations $f_G=f'(G)$ and $f_{GG}=f''(G)$. We shall assume throughout a spatially-flat FLRW universe. Then, the field equations give the FLRW equations, with the form

$$0=-\frac{3}{\kappa^2}H^2+Gf_G-f(G)-24\dot{G}H^3f_{GG}+\rho_m\,,$$

$$0=8H^2\ddot{f}_G+16H(\dot{H}+H^2)\dot{f}_G+\frac{1}{\kappa^2}(2\dot{H}+3H^2)+f-Gf_G+p_m\,. \tag{7.4}$$

The Gauss-Bonnet invariant G and the Ricci scalar R can be defined as functions of the Hubble parameter as

$$G=24(\dot{H}H^2+H^4),\quad R=6(\dot{H}+2H^2). \tag{7.5}$$

Let us now rewrite Eq. (7.4) by using a new variable, $N=\ln\frac{a}{a_0}=-\ln(1+z)$, i.e. the number of e-foldings, instead of the cosmological time t , where z is the redshift. The following expressions are then easily obtained

$$a=a_0e^N,\quad H=\dot{N}=\frac{dN}{dt}\,,\quad \frac{d}{dt}=H\frac{d}{dN}\,,\quad \frac{d^2}{dt^2}=H^2\frac{d^2}{dN^2}+HH'\frac{d}{dN}\,,\quad H'=\frac{dH}{dN}\,. \tag{7.6}$$

Eq. (7.4) can thus be expressed as follows

$$0=-\frac{3}{\kappa^2}H^2+24H^3(H'+H)f_G-f-576H^6\left(HH''+3H'^2+4HH'\right)f_{GG}+\rho_m\,, \tag{7.7}$$

where G and R are now

$$G=24(H^3H'+H^4)\,,\quad \dot{G}=24(H^4H''+3H^3H'^2+4H^4H')\,,\quad R=6(HH'+2H^2)\,. \tag{7.8}$$

By introducing a new function x as $x=H^2$, we have

$$H'=\frac{1}{2}x^{-1/2}x'\,,\quad H''=-\frac{1}{4}x^{-3/2}x'^2+\frac{1}{2}x^{-1/2}x''\,. \tag{7.9}$$

Hence, Eq. (7.7) takes the form

$$0=-\frac{3}{\kappa^2}x+12x(x'+2x)f_G-f-24^2x\left[\frac{1}{2}x^2x''+\frac{1}{2}xx'^2+2x^2x'\right]f_{GG}+\rho_m\,, \tag{7.10}$$

where we have used the expressions

$$G=12xx'+24x^2,\quad \dot{G}=12x^{-1/2}[x^2x''+xx'^2+4x^2x']\quad\text{and}\quad R=3x'+12x. \tag{7.11}$$

Then, by using the above reconstruction method, any cosmological solution can be achieved, by introducing the given Hubble parameter in the FLRW equations, which leads to the corresponding Gauss-Bonnet action.

7.2 Reconstructing Λ CDM model in $R + f(G)$ gravity

We are now interested to reconstruct Λ CDM solution in $R + f(G)$ gravity for different kind of matter contributions. We show that in GB gravity there is no need of a cc. The cosmological models coming from the different versions of modified GB gravity considered will be carefully investigated with the help of several particular examples where calculations can be carried out explicitly. For the Λ CDM model, the Hubble rate is given by

$$H^2 = \frac{\Lambda}{3} + \frac{\rho_0}{3a^3}, \tag{7.12}$$

where ρ_0 is the matter density (which consists of barionic matter and cold dark matter) and Λ is the cosmological constant. In the rest of this section we put $\kappa^2 = 1$.

For the Λ CDM model, described by the Hubble parameter (7.12), we can write the derivatives of the scale factor as well as the Hubble parameter in the following useful way:

$$\dot{a} = \sqrt{\frac{\Lambda a^2}{3} + \frac{\rho_0}{3a}} \ , \quad \ddot{a} = \frac{2\Lambda a^3 - \rho_0}{6a^2} \ ,$$

$$\dot{H} = -\frac{\rho_0}{2a^3} = \frac{3}{2}\left(\frac{\Lambda}{3} - H^2\right) \ , \quad \ddot{H} = \frac{3\rho_0}{2a^3}\sqrt{\frac{\Lambda}{3} + \frac{\rho_0}{3a^3}} = \frac{9}{2}\left(H^2 - \frac{\Lambda}{3}\right) H \ . \tag{7.13}$$

Using these formulas we get

$$R = 4\Lambda + \frac{\rho_0}{a^3} \ , \quad G = 24\left(\frac{\rho_0}{3a^3} + \frac{\Lambda}{3}\right)\left(\frac{\Lambda}{3} - \frac{\rho_0}{6a^3}\right) \ ,$$

$$\dot{R} = \frac{-3\rho_0}{a^3}\sqrt{\frac{\rho_0}{3a^3} + \frac{\Lambda}{3}} \ , \quad \dot{G} = \frac{4\rho_0}{a^3}\left(\frac{2\rho_0}{a^3} - \Lambda\right)\sqrt{\frac{\rho_0}{3a^3} + \frac{\Lambda}{3}} \ . \tag{7.14}$$

Then, the following relation between R and G holds:

$$G = -\frac{4}{3}(R^2 - 9\Lambda R + 18\Lambda^2) \ . \tag{7.15}$$

Let us recall that $x = H^2$. Then, some of the above formulas take the form:

$$\dot{H} = \frac{1}{2}(\Lambda - 3x) \quad \ddot{H} = \frac{3}{2}(3x - \Lambda)\sqrt{x} \ , \quad R = 3(\Lambda + x) \ ,$$

$$G = 12x(\Lambda - x) \ , \quad \dot{R} = 3(\Lambda - 3x)\sqrt{x} \ , \quad \dot{G} = 12(3x - \Lambda)(2x - \Lambda)\sqrt{x} \ . \tag{7.16}$$

Note that the variable x can be expressed in terms of R or G as

$$x = \frac{R}{3} - \Lambda \quad \text{or} \quad x = \frac{3\Lambda \pm \sqrt{9\Lambda^2 - 3G}}{6} \ , \tag{7.17}$$

respectively. The above formulas will be useful to reconstruct the Λ CDM model as well as other cosmological solutions in the context of Gauss-Bonnet gravity, as it is shown below.

We write the first Friedmann equation (7.4) in the form

$$0 = -3H^2 + 12H^2(\Lambda - H^2)f_G - f - 288H^4(3H^2 - \Lambda)(2H^2 - \Lambda)f_{GG} + \rho_m \ , \tag{7.18}$$

or

$$0 = -3x + 12x(\Lambda - x)f_G - f - 288x^2(3x - \Lambda)(2x - \Lambda)f_{GG} + \rho_m \ . \tag{7.19}$$

For further algebra more convenient is the following form of this equation

$$0 = (\rho_m - 3x - f)(\Lambda - 2x) + [48x^2(3x - \Lambda) + x(\Lambda - x)]f_x + 24x^2(3x - \Lambda)(\Lambda - 2x)f_{xx} \ . \tag{7.20}$$

Now we wish to construct some particular exact solutions of this equation.

Case I: Absence of matter, $\rho_m = 0$

First of all, let us consider the simple case in absence of matter, $\rho_m = 0$. Then, the equation (7.20) takes the form

$$0 = -(3x+f)(\Lambda-2x) + [48x^2(3x-\Lambda) + x(\Lambda-x)]f_x + 24x^2(3x-\Lambda)(\Lambda-2x)f_{xx} \ . \tag{7.21}$$

We can analyze the cases where the cosmological constant term in the solution (7.12) vanishes and where it is non-zero.

- Let $\Lambda = 0$. Then Eq. (7.21) reads as

  $$0 = -2(3x+f) - x(144x-1)f_x + 144x^3 f_{xx} \ . \tag{7.22}$$

  The general solution of (7.22) is given by

  $$f(x) = C_2x^2 + C_1x(144x-1)e^{\frac{1}{144x}} + v_1(x) \ , \tag{7.23}$$

  where

  $$v_1(x) = -864x\left[\frac{1}{144x} + x\ln x + \left(\frac{1}{144x} - x\right)Ei\left(1, \frac{1}{144x}\right)e^{\frac{1}{144x}}\right] \ . \tag{7.24}$$

  Here

  $$Ei(a,z) = z^{a-1}\Gamma(1-a,z) = \int_1^\infty e^{-zs}s^{-a}ds \ . \tag{7.25}$$

  Then, the solution that reproduces, basically a power-law expansion, is found.

- Let $\Lambda \neq 0$. Then Eq. (7.21) has a complex solution, which has no physical meaning as it gives a complex action.

Hence, it appears that the Λ CDM model (7.12) can not be reproduced by $R+f(G)$ gravity in the absence of matter. The only solution found, restricted to $\Lambda = 0$, does not produce an accelerating expansion.

Case II: $\rho_m \neq 0$ and $\Lambda = 0$

We now explore the case when some kind of matter with a particular EoS is present in the Universe, but with no cosmological constant term in the Hubble parameter described in (7.12). We explore several examples where different kind of matter contributions are considered.

Example 1

Let us now consider the case when $\Lambda = 0$ and the evolution of the matter density behaves as

$$\rho_m = 3H^2 = 3x. \tag{7.26}$$

In this case the modified Friedmann equation (7.4) reads as

$$0 = 2f + x(144x-1)f_x - 144x^3 f_{xx}. \tag{7.27}$$

The general solution of the equation (7.27) is given by

$$f(x) = C_1x^2 + C_2x(144x-1)e^{\frac{1}{144x}}. \tag{7.28}$$

This function reproduces the solution (7.12) under the conditions imposed above.

Example 2

Now we consider a more general case, where the energy density is given by,

$$\rho_m = u(x) \ , \tag{7.29}$$

where $u(x)$ is some function of x. In this case the modified Friedmann equation (7.4) reads as

$$0 = 2[3x - u(x) + f] + x(144x - 1)f_x - 144x^3 f_{xx} \ . \tag{7.30}$$

Its general solution is

$$f(x) = C_1 x^2 + C_2 x(144x - 1)e^{\frac{1}{144x}} + v_2(x) \ , \tag{7.31}$$

with

$$v_2(x) = 288x \left[\left(x - \frac{1}{144} \right) e^{\frac{1}{144x}} J_1 - \frac{x}{144} J_2 \right] \ , \tag{7.32}$$

where

$$J_1 = \int \frac{3x - u}{x^2} e^{\frac{1}{144x}} dx, \quad J_2 = \int \frac{(3x-u)(144x-1)}{x^3} dx \ . \tag{7.33}$$

Then, the solution (7.31) gives the function of the Gauss-Bonnet invariant that reproduces this model for any kind of EoS matter fluid.

Case III: $\rho_m \neq 0$ and $\Lambda \neq 0$

Let us now explore the most general case for the solution (7.12) in $R+f(G)$ gravity with a non vanishing matter fluid with a given EoS parameter.

We consider a Universe filled with a pressureless fluid. By the energy conservation equation, the energy desity can be written as

$$\rho_m = 3x + \beta \, . \tag{7.34}$$

Then Eq. (7.4) takes the form

$$0 = (\beta - f)(\Lambda - 2x) + [48x^2(3x - \Lambda) + x(\Lambda - x)]f_x + 24x^2(3x - \Lambda)(\Lambda - 2x)f_{xx} \tag{7.35}$$

and has the following particular solution:

$$f(x) = \gamma x^2 - \gamma \Lambda x + \beta \, . \tag{7.36}$$

If $\Lambda = 0$, then $f(x) = \gamma x^2 + \beta$. Also if $\gamma = 0$, then the solution takes the form $f = \beta$, which corresponds to the cosmological constant. Note that if $\beta = -\Lambda$ then

$$\rho_m = \frac{\rho_{03}}{a^3} = 3x - \Lambda, \quad f(x) = \gamma x^2 - \gamma \Lambda x - \Lambda \ . \tag{7.37}$$

This gives a solution where the cosmological constant is corrected by the contribution from $f(G)$, what may resolve the cosmological constant problem.

7.3 The $F(R,G)$ model

Let us now consider a more general model for a class of modified Gauss-Bonnet gravity. This can be described by the following action

$$S = \int d^4x\sqrt{-g}[\frac{1}{2\kappa^2}F(R,G) + L_m]. \tag{7.38}$$

Varying over $g_{\mu\nu}$ the gravity field equations are obtained,

$$\begin{aligned} 0 = \kappa^2 T^{\mu\nu} + \frac{1}{2}g^{\mu\nu}F(G) - 2F_G R R^{\mu\nu} + 4F_G R^{\mu}_{\rho} R^{\nu\rho} \\ -2F_G R^{\mu\rho\sigma\tau} R^{\nu}_{\rho\sigma\tau} - 4F_G R^{\mu\rho\sigma\nu} R_{\rho\sigma} + 2(\nabla^\mu\nabla^\nu F_G)R - 2g^{\mu\nu}(\nabla^2 F_G)R \\ -4(\nabla_\rho\nabla^\mu F_G)R^{\nu\rho} - 4(\nabla_\rho\nabla^\nu F_G)R^{\mu\rho} + 4(\nabla^2 F_G)R^{\mu\nu} + 4g^{\mu\nu}(\nabla_\rho\nabla_\sigma F_G)R^{\rho\sigma} \\ -4(\nabla_\rho\nabla_\sigma F_G)R^{\mu\rho\nu\sigma} - F_G R^{\mu\nu} + \nabla^\mu\nabla^\nu F_R - g^{\mu\nu}\nabla^2 F_R. \end{aligned} \tag{7.39}$$

In the case of a flat FLRW Universe, the first FLRW equation yields

$$0 = \frac{1}{2}(GF_G - F - 24H^3 F_{Gt}) + 3(\dot{H} + H^2)F_R - 3HF_{Rt} + \kappa^2\rho_m. \tag{7.40}$$

And from here, using the techniques developed in the previous section, it is plain that explicit $F(R,G)$ functions can be reconstructed for given cosmological solutions.

De Sitter Solutions

As well known, the de Sitter solution is one of the most important cosmological solutions nowadays, since the current epoch has been observed to have an expansion that behaves approximately as de Sitter. This solution is described by an exponential expansion of the scale factor, which gives a constant Hubble parameter $H(t) = H_0$. By inserting it in the Friedmann equation (7.40), one finds that any kind of $F(R,G)$ function can possibly admit de Sitter solutions, with the proviso that the following algebraic equation has positive roots for H_0

$$0 = \frac{1}{2}(G_0 F_G(G_0) - F(G_0,R_0)) + 3H_0^2 F_R(R_0)\ , \tag{7.41}$$

being $R_0 = 12H_0^2$ and $G_0 = 24H_0^4$; we have here neglected the contribution of matter for simplicity. As it was pointed out for the case of modified $F(R)$ gravity, the de Sitter points are critical points for the Friedmann equations, what could explain the current acceleration phase as well as the inflationary epoch. This explanation can be extended to the action (7.38), so that any kind of function $F(R,G)$ with positive real roots for the equation (7.41) could in fact explain the acceleration epochs of the Universe in exactly the same way a cosmological constant does.

Phantom dark energy

Let us now explore the cosmic evolution described by $H = \mathrm{e}^{mN}$ in the context of the action (7.38). This solution reproduces a phantom behavior, i.e. a superaccelerated expansion that, according to recent observations, our Universe could be in—or either close to cross the phantom barrier. We can now proceed

with the reconstruction method, as explicitly shown in the section above, and a $F(R,G)$ function will be reconstructed. For simplicity, we consider the following subfamily of functions

$$F(R,G) = f_1(G) + f_2(R) \ . \tag{7.42}$$

Correspondingly, the Friedmann equation (7.40) can be split into two equations, as

$$\begin{aligned} 0 &= -24H^3\dot{G}f_{1GG} + Gf_{1G} - f_1 \ , \\ 0 &= -3H\dot{R}f_{2RR} + 3(\dot{H}+H^2)f_{2R} - \frac{1}{2}f_2 + \kappa^2\rho_m \end{aligned} \tag{7.43}$$

For this example, the Ricci scalar and the Gauss-Bonnet terms take the following form,

$$\begin{aligned} G &= 24(m+1)e^{4mN} = 24(m+1)H^4, \\ R &= 6(m+2)e^{2mN} = 6(m+2)H^2 \ . \end{aligned} \tag{7.44}$$

Hence, the first equation in (7.43) can be written in terms of G, as

$$G^2 f_{1GG} - \frac{m+1}{4m}Gf_{1G} + \frac{m+1}{4m}f_1 = 0 \ . \tag{7.45}$$

This is an Euler equation, easy to solve, and yields

$$f_1(G) = C_1 G^{1+\frac{m+1}{4m}} + C_2 \ , \tag{7.46}$$

where $C_{1,2}$ are integration constants. In the same way, for the case being considered here, the second equation in (7.43), for R, takes the form

$$R^2 f_{2RR} - \frac{m+1}{2m}Rf_{2R} + \frac{m+2}{2m}f_2 - \frac{\kappa^2(m+2)}{m}\rho_m = 0 \ . \tag{7.47}$$

In absence of matter ($\rho_m = 0$) this is also an Euler equation, with solution

$$\begin{aligned} & f_2(R) = k_1 R^{\mu_+} + k_2 R^{\mu_-} \\ \text{where} \quad & \mu_\pm = \frac{1+\frac{m+1}{2m} \pm \sqrt{1+\frac{(m+1)^2}{4m^2} - \frac{m+3}{m}}}{2} \ , \end{aligned} \tag{7.48}$$

and $k_{1,2}$ are integration constants. Then, the complete function F(R,G), given in (7.42), is reconstructed (in absence of matter) yielding the solutions (7.46) and (7.48). The theory (3.12) belongs to the class of models with positive and negative powers of the curvature introduced in [219].

Let us now consider the case where matter is included. From the energy conservation equation $\dot{\rho_m} + 3H(1+w)\rho_m = 0$ we have that, for a perfect fluid with constant EoS, $p_m = w_m\rho_m$, the solution is given by $\rho_m = \rho_0 e^{-3(1+w_m)N}$. By inserting the expression for R (7.44) into this solution, we get

$$\rho_m = \rho_{m0} = \left(\frac{R}{6(m+2)}\right)^{-\frac{3(1+w_m)}{2m}} \ . \tag{7.49}$$

In such case the general solution for f_2 is given by

$$\begin{aligned} & f_2(R) = k_1 R^{\mu_+} + k_2 R^{\mu_-} + kR^A \text{ , where,} \\ & k = \kappa^2 \frac{\rho_{m0}(6(m+1))^{-A}}{A(A-1) - 1/2m} \quad \text{and} \quad A = -\frac{3(1+w_m)}{2m} \ . \end{aligned} \tag{7.50}$$

Hence, we see that the solution for the Hubble parameter $H = \mathrm{e}^{mN}$ can be easily recovered in the context of modified Gauss-Bonnet gravity. Nevertheless, it seems clear that, for more complex examples, one may not be able to solve the corresponding equations analytically and numerical analysis could be required..

7.4 Cosmological solutions in pure $f(G)$ gravity

We have studied so far a theory described by the action (7.1), which is given by the usual Hilbert-Einstein term plus a function of the Gauss-Bonnet invariant, that is assumed to become important in the dark energy epoch. In this section, we are interested to investigated some important cosmic solutions in the frame of a theory described only by the Gauss-Bonnet invariant, and whose action is given by

$$S = \int d^4x\sqrt{-g}\left[f(G) + L_m\right] \ . \tag{7.51}$$

In this case, the FLRW equations are:

$$\begin{aligned} 0 &= Gf_G - f - 24\dot{G}H^3 f_{GG} + \rho_m \\ 0 &= 8H^2\ddot{f}_G + 16H(\dot{H} + H^2)\dot{f}_G + f - Gf_G + p_m \ . \end{aligned} \tag{7.52}$$

We are interested to explore some important solutions from the cosmological point of view, as de Sitter and power law expansions. De Sitter solutions can be easily checked for a given model, as it was shown above. We can explore the de Sitter points admitted by a general $f(G)$ by introducing the solution $H(t) = H_0$ in the first FLRW equation given in (7.52), which yields

$$0 = G_0 f_G(G_0) - f(G_0) \ . \tag{7.53}$$

Here, $G_0 = 24H_0^2$ and we have ignored the contribution of matter. Then, we have reduced the differential equation (7.52) to an algebraic equation that can be resolved by specifying a function $f(G)$. The de Sitter points are given by the positive roots of this equation, which could explain not just the late-time accelerated epoch but also the inflationary epoch. The stability of these solutions has to be studied in order to achieve a grateful exit in the case of inflation, and future predictions for the current cosmic acceleration.

Power law solutions

We are now interested to explore power law solutions for a theory described by the action (7.51). This kind of solutions are very important during the cosmic history as the matter/radiation epochs are described by power law expansions, as well as the possible phantom epoch, which can be seen as a special type of these solutions. Let us start by studying a Hubble parameter given by

$$H(t) = \frac{\alpha}{t} \rightarrow a(t) \sim t^{\alpha} \ , \tag{7.54}$$

where we take $\alpha > 1$. Then, by introducing the solution (7.54) into the first FLRW equation (7.52), it yields the differential equation

$$0 = -f(G) + Gf_G + \frac{4G^2}{\alpha - 1} f_{GG} \ , \tag{7.55}$$

where we have neglected any contribution of matter for simplicity. The equation (7.55) is a type of Euler equation, whose solution is

$$f(G) = C_1 G + C_2 G^{\frac{1-\alpha}{4}} \ . \tag{7.56}$$

Thus, we have shown that power-law solutions of the type (7.54) correspond to actions with powers on the Gauss-Bonnet invariant, in a similar way as in $f(R)$ gravity, where power-law solutions correspond to an action with powers on the scalar curvature, R, as it was shown above.

Let us now explore another kind of power-law solutions, where the Universe enters a phantom phase and ends in a Big Rip singularity. This general class of Hubble parameters may be written as

$$H(t) = \frac{\alpha}{t_s - t} \ , \tag{7.57}$$

where t_s is the so-called Rip time, i.e. the time when the future singularity will take place. By inserting the solution (7.57) into the first FLRW equation (7.52), the equation yields

$$0 = -f(G) + Gf_G(G) - \frac{4\alpha^2 G^2}{1+\alpha} \ , \tag{7.58}$$

which is also a Euler equation, whose solution is given by,

$$f(G) = C_1 G + C_2 G^{\frac{1+\alpha}{4\alpha^2}} \ . \tag{7.59}$$

Thus, we have shown that power law solution of the type radiation/matter dominated epochs on one side and phantom epochs on the other, are well reproduced in pure $f(G)$ gravity, in a similar way as it is in $f(R)$ gravity.

7.5 Conclusions

We have explored in this chapter several cosmological solutions in the frame of Gauss-Bonnet gravity, considering specially the case of an action composed of the Hilbert-Einstein action plus a function on the Gauss-Bonnet invariant. Also pure $f(G)$ gravity has been considered, as well as the possibility of the implication of inhomogeneous terms in the EoS of a perfect fluid, which could contribute together with modified gravity to the late-time acceleration. We have shown that the ΛCDM model can well be explained in this kind of theories, which may give an explanation to the cosmological constant problem as the modified gravity terms may act relaxing the vacuum energy density. Other kinds of solutions in $f(G)$ gravity have been reconstructed. It has been shown that $f(G)$ gravity could explain the dark energy epoch whatever the nature of its EoS, of type quintessence or phantom, and even the inflationary phase. More complex cosmological solutions would require numerical analysis, but our analysis of a few simple cases has already shown that $f(G)$ gravity accounts for the accelerated epochs and may contribute during the radiation/matter dominated eras, and it may explain also the dark matter contributions to the cosmological evolution, what will be explored in future works. This kind of modified gravity models which reproduce dark energy and inflation, can be modeled as an inhomogeneous fluid with a dynamical equation of state, what would be distinguished from other models with a static EoS. Even as perturbations in modified gravity behave different than in General Relativity, it could give a signature of the presence of higher order terms in the gravity action, as the Gauss-Bonnet invariant, when structure formation is studied and simulations are performed, what should be explored in the future.

Part III

On Hořava-Lifshitz gravity and its extension to more general actions in cosmology

Chapter 8

Unifying inflation with dark energy in modified F(R) Hořava-Lifshitz gravity

[1]The Hořava-Lifshitz quantum gravity [174] has been conjectured to be renormalizable in four dimensions, at the price of explicitly breaking Lorentz invariance. Its generalization to an $F(R)$-formulation, which seems to be also renormalizable in $3+1$ dimensions, has been considered in Refs. [90, 100], where the Hamiltonian structure and FRW cosmology, in a power-law theory, have been investigated for such modified $F(R)$ Hořava-Lifshitz gravity. It was also conjectured there that it sustains, in principle, the possibility of a unified description of early-time inflation and the dark energy epochs.

The purpose of the present chapter is to show a realistic non-linear $F(R)$ gravity in the Hořava-Lifshitz formulation, with the aim to understand if such theory is in fact directly able to predict in a natural way the unification of the two acceleration eras, similarly as it is done in the convenient version. It will be here shown that, for a special choice of parameters, the FRW equations do coincide with the ones for the related, convenient $F(R)$ gravity. This means, in particular, that the cosmological history of such Hořava-Lifshitz $F(R)$ gravity will be just the same as for its convenient version (whereas black hole solutions are generically speaking different). For the general version of the theory the situation turns out to be more complicated. Nevertheless, the unification of inflation with dark energy is still possible and all local tests can also be passed, as we will prove.

8.1 Modified $F(R)$ Hořava-Lifshitz gravity

In this section, modified Hořava-Lifshitz $F(R)$ gravity is briefly reviewed [90, 100]. We start by writing a general metric in the so-called ADM decomposition in a $3+1$ spacetime (for more details see [14, 207] and references therein),

$$ds^2 = -N^2 dt^2 + g_{ij}^{(3)}(dx^i + N^i dt)(dx^j + N^j dt)\,, \tag{8.1}$$

[1]This Chapter is based on: [135]

where $i, j = 1, 2, 3$, N is the so-called lapse variable, and N^i is the shift 3-vector. In standard general relativity (GR), the Ricci scalar can be written in terms of this metric, and yields

$$R = K_{ij}K^{ij} - K^2 + R^{(3)} + 2\nabla_\mu(n^\mu\nabla_\nu n^\nu - n^\nu\nabla_\nu n^\mu)\,, \tag{8.2}$$

here $K = g^{ij}K_{ij}$, K_{ij} is the extrinsic curvature, $R^{(3)}$ is the spatial scalar curvature, and n^μ a unit vector perpendicular to a hypersurface of constant time. The extrinsic curvature K_{ij} is defined as

$$K_{ij} = \frac{1}{2N}\left(\dot{g}^{(3)}_{ij} - \nabla^{(3)}_i N_j - \nabla^{(3)}_j N_i\right)\,. \tag{8.3}$$

In the original model [174], the lapse variable N is taken to be just time-dependent, so that the projectability condition holds and by using the foliation-preserving diffeomorphisms (8.6), it can be fixed to be $N = 1$. As pointed out in [32], imposing the projectability condition may cause problems with Newton's law in the Hořava gravity. On the other hand, Hamiltonian analysis shows that the non-projectable $F(R)$-model is inconsistent[101]). For the non-projectable case, the Newton law could be restored (while keeping stability) by the "healthy" extension of the original Hořava gravity of Ref. [32].

The action for standard $F(R)$ gravity can be written as

$$S = \int d^4x\sqrt{g^{(3)}}NF(R)\,. \tag{8.4}$$

Gravity of Ref. [174] is assumed to have different scaling properties of the space and time coordinates

$$x^i = bx^i\,, \quad t = b^z t\,, \tag{8.5}$$

where z is a dynamical critical exponent that renders the theory renormalizable for $z = 3$ in $3+1$ spacetime dimensions [174] (For a proposal of covariant renormalizable gravity with dynamical Lorentz symmetry breaking, see GR is recovered when $z = 1$. The scaling properties (8.5) render the theory invariant only under the so-called foliation-preserving diffeomorphisms:

$$\delta x^i = \zeta(x^i, t)\,, \quad \delta t = f(t)\,. \tag{8.6}$$

It has been pointed that, in the IR limit, the full diffeomorphisms are recovered, although the mechanism for this transition is not physically clear. The action considered here was introduced in Ref. [100],

$$S = \frac{1}{2\kappa^2}\int dtd^3x\sqrt{g^{(3)}}NF(\tilde{R})\,, \quad \tilde{R} = K_{ij}K^{ij} - \lambda K^2 + R^{(3)} + 2\mu\nabla_\mu(n^\mu\nabla_\nu n^\nu - n^\nu\nabla_\nu n^\mu) - L^{(3)}(g^{(3)}_{ij})\,, \tag{8.7}$$

where κ is the dimensionless gravitational coupling, and where, two new constants λ and μ appear, which account for the violation of the full diffeomorphism transformations. A degenerate version of the above $F(R)$-theory with $\mu = 0$ has been proposed and studied in Ref. [188, 190]. Note that in the original Hořava gravity theory [174], the third term in the expression for $\tilde{R}$ can be omitted, as it becomes a total derivative. The term $L^{(3)}(g^{(3)}_{ij})$ is chosen to be [174]

$$L^{(3)}(g^{(3)}_{ij}) = E^{ij}G_{ijkl}E^{kl}\,, \tag{8.8}$$

where G_{ijkl} is the generalized De Witt metric, namely

$$G^{ijkl} = \frac{1}{2}\left(g^{ik}g^{jl} + g^{il}g^{jk}\right) - \lambda g^{ij}g^{kl}\,. \tag{8.9}$$

In Ref. [174], the expression for E_{ij} is constructed to satisfy the "detailed balance principle" in order to restrict the number of free parameters of the theory. This is defined through variation of an action

$$\sqrt{g^{(3)}}E^{ij} = \frac{\delta W[g_{kl}]}{\delta g_{ij}}\,, \tag{8.10}$$

where the form of $W[g_{kl}]$ is given in Ref. [173] for $z = 2$ and $z = 3$. Other forms for $L^{(3)}(g_{ij}^{(3)})$ have been suggested that abandons the detailed balance condition but still render the theory power-counting renormalizable (see Ref. [90]).

We are interested in the study of (accelerating) cosmological solutions for the theory described by action (8.7). Spatially-flat FRW metric is assumed, which is written in the ADM decomposition as,

$$ds^2 = -N^2 dt^2 + a^2(t) \sum_{i=1}^{3} \left(dx^i\right)^2\,. \tag{8.11}$$

If we also assume the projectability condition, N can be taken to be just time-dependent and, by using the foliation-preserving diffeomorphisms (8.6), it can be fixed to be unity, $N = 1$. When we do not assume the projectability condition, N depends on both the time and spatial coordinates, first. Then, just as an assumption of the solution, N is taken to be unity.

For the metric (8.11), the scalar $\tilde{R}$ is given by

$$\tilde{R} = \frac{3(1-3\lambda+6\mu)H^2}{N^2} + \frac{6\mu}{N}\frac{d}{dt}\left(\frac{H}{N}\right)\,. \tag{8.12}$$

For the action (8.7), and assuming the FRW metric (8.12), the second FRW equation can be obtained by varying the action with respect to the spatial metric $g_{ij}^{(3)}$, which yields

$$0 = F(\tilde{R}) - 2(1-3\lambda+3\mu)\left(\dot{H}+3H^2\right)F'(\tilde{R}) - 2(1-3\lambda)\dot{\tilde{R}}F''(\tilde{R}) + 2\mu\left(\dot{\tilde{R}}^2 F^{(3)}(\tilde{R}) + \ddot{\tilde{R}}F''(\tilde{R})\right) + \kappa^2 p_m\,, \tag{8.13}$$

here $\kappa^2 = 16\pi G$, p_m is the pressure of a perfect fluid that fills the Universe, and $N = 1$. Note that this equation becomes the usual second FRW equation for convenient $F(R)$ gravity (8.4), by setting the constants $\lambda = \mu = 1$. When we assume the projectability condition, variation over N of the action (8.7) yields the following global constraint

$$0 = \int d^3x \left[F(\tilde{R}) - 6(1-3\lambda+3\mu)H^2 - 6\mu\dot{H} + 6\mu H\dot{\tilde{R}}F''(\tilde{R}) - \kappa^2\rho_m\right]\,. \tag{8.14}$$

Now, using the ordinary conservation equation for the matter fluid $\dot{\rho}_m + 3H(\rho_m + p_m) = 0$, and integrating Eq. (8.13),

$$0 = F(\tilde{R}) - 6\left[(1-3\lambda+3\mu)H^2 + \mu\dot{H}\right]F'(\tilde{R}) + 6\mu H\dot{\tilde{R}}F''(\tilde{R}) - \kappa^2\rho_m - \frac{C}{a^3}\,, \tag{8.15}$$

where C is an integration constant, taken to be zero, according to the constraint equation (8.14). If we do not assume the projectability condition, we can directly obtain (8.15), which corresponds to the first FRW equation, by variation over N. Hence, starting from a given $F(\tilde{R})$ function, and solving Eqs. (8.13) and (8.14), a cosmological solution can be obtained.

8.2 Reconstructing FRW cosmology in $F(R)$ Hořava-Lifshitz gravity

To start, let us analyze the simple model $F(\tilde{R}) = \tilde{R}$, which cosmology was studied in [4, 16, 37, 34, 41, 68, 69, 92, 154, 178, 206, 211, 214, 238, 251, 252, 276, 280, 282, 289, 300, 301] (for a complete analysis of cosmological perturbations, see [158]). In such a case, the FRW equations look similar to GR,

$$H^2 = \frac{\kappa^2}{3(3\lambda - 1)}\rho_m\,, \quad \dot{H} = -\frac{\kappa^2}{2(3\lambda - 1)}(\rho_m + p_m)\,, \tag{8.16}$$

where, for $\lambda \to 1$, the standard FRW equations are recovered. Note that the constant μ is now irrelevant because, as pointed out above, the term in front of μ in (8.7) becomes a total derivative. For such theory, one has to introduce a dark energy source as well as an inflaton field, in order to reproduce the cosmic and inflationary accelerated epochs, respectively. It is also important to note that, for this case, the coupling constant is restricted to be $\lambda > 1/3$, otherwise Eqs. (8.16) become inconsistent. It seems reasonable to think that, for the current epoch, where $\tilde{R}$ has a small value, the IR limit of the theory is satisfied $\lambda \sim 1$, but for the inflationary epoch, when the scalar curvature $\tilde{R}$ goes to infinity, λ will take a different value. It has been realized that, for $\lambda = 1/3$, the theory develops an anisotropic Weyl invariance (see [174]), and thus it takes an special role, although for the present model this value is not allowed.

We now discuss some cosmological solutions of $F(\tilde{R})$ Hořava-Lifshitz gravity. The first FRW equation, given by (8.15) with $C = 0$, can be rewritten as a function of the number of e-foldings $\eta = \ln \frac{a}{a_0}$, instead of the usual time t. This technique has been developed in Chapter 5 for convenient $F(R)$ gravity, where it was shown that any $F(R)$ theory can be reconstructed for a given cosmological solution. Here, we extend such formalism to the Hořava-Lifshitz $F(R)$ gravity. Since $\frac{d}{dt} = H\frac{d}{d\eta}$ and $\frac{d^2}{d\eta^2} = H^2\frac{d^2}{d\eta^2} + H\frac{dH}{d\eta}\frac{d}{d\eta}$, the first FRW equation (8.15) is rewritten as

$$0 = F(\tilde{R}) - 6\left[(1-3\lambda+3\mu)H^2 + \mu HH'\right]\frac{dF(\tilde{R})}{d\tilde{R}} + 36\mu H^2\left[(1-3\lambda+6\mu)HH' + \mu H'^2 + \mu H'' H\right]\frac{d^2F(\tilde{R})}{d^2\tilde{R}} - \kappa^2\rho_m\,, \tag{8.17}$$

where the primes denote derivatives with respect to η. Thus, in this case there is no restriction on the values of λ or μ. By using the energy conservation equation, and assuming a perfect fluid with equation of state (EoS) $p_m = w_m\rho_m$, the energy density yields

$$\rho_m = \rho_0 a^{-3(1+w_m)} = \rho_0 a_0^{-3(1+w_m)}\mathrm{e}^{-3(1+w_m)\eta}\,. \tag{8.18}$$

As the Hubble parameter can be written as a function of the number of e-foldings, $H = H(\eta)$, the scalar curvature in (8.12) takes the form

$$\tilde{R} = 3(1-3\lambda+6\mu)H^2 + 6\mu HH'\,, \tag{8.19}$$

which can be solved with respect to η as $\eta = \eta(\tilde{R})$, and one gets an expression (8.17) that gives an equation on $F(\tilde{R})$ with the variable $\tilde{R}$. This can be simplified a bit by writing $G(\eta) = H^2$ instead of the Hubble parameter. In such case, the differential equation (8.17) yields

$$0 = F(\tilde{R}) - 6\left[(1-3\lambda+3\mu)G + \frac{\mu}{2}G'\right]\frac{dF(\tilde{R})}{d\tilde{R}} + 18\mu\left[(1-3\lambda+6\mu)GG' + \mu GG''\right]\frac{d^2F(\tilde{R})}{d^2\tilde{R}} - \kappa^2\rho_0 a_0^{-3(1+w)}\mathrm{e}^{-3(1+w)\eta}\,, \tag{8.20}$$

and the scalar curvature is now written as $\tilde{R} = 3(1-3\lambda+6\mu)G + 3\mu G'$. Hence, for a given cosmological solution $H^2 = G(\eta)$, one can resolve Eq. (8.20), and the $F(\tilde{R})$ that reproduces such solution is obtained.

As an example, we consider the Hubble parameter that reproduces the Λ CDM epoch. It is expressed as

$$H^2 = G(\eta) = H_0^2 + \frac{\kappa^2}{3}\rho_0 a^{-3} = H_0^2 + \frac{\kappa^2}{3}\rho_0 a_0^{-3}\mathrm{e}^{-3\eta}\,. \tag{8.21}$$

where H_0 and ρ_0 are constant. In General Relativity, the terms on the rhs of Eq. (8.21) correspond to an effective cosmological constant $\Lambda = 3H_0^2$ and to cold dark matter with EoS parameter $w = 0$. The corresponding $F(\tilde{R})$ can be reconstructed by following the same steps as described above. Using the expression for the scalar curvature $\tilde{R} = 3(1-3\lambda+6\mu)G + 3\mu G'$, the relation between $\tilde{R}$ and η is obtained,

$$\mathrm{e}^{-3\eta} = \frac{\tilde{R} - 3(1-3\lambda+6\mu)H_0^2}{3k(1+3(\mu-\lambda))}\,, \tag{8.22}$$

where $k = \frac{\kappa^2}{3}\rho_0 a_0^{-3}$. Then, substituting (8.21) and (8.22) into Eq. (8.20), one gets the differential expression

$$\begin{aligned} 0 \;=\;& (1-3\lambda+3\mu)F(\tilde{R}) - 2\left(1-3\lambda+\frac{3}{2}\mu\right)\tilde{R} + 9\mu(1-3\lambda)H_0^2\frac{dF(\tilde{R})}{d\tilde{R}} \\ & -6\mu(\tilde{R}-9\mu H_0^2)(\tilde{R}-3H_0^2(1-3\lambda+6\mu))\frac{d^2F(\tilde{R})}{d^2\tilde{R}} - R - 3(1-3\lambda+6\mu)H_0^2\,, \end{aligned} \tag{8.23}$$

where, for simplicity, we have considered a pressureless fluid $w = 0$ in Eq. (8.20). Performing the change of variable $x = \frac{\tilde{R}-9\mu H_0^2}{3H_0^2(1+3(\mu-\lambda))}$, the homogeneous part of Eq. (8.23) can be easily identified as an hypergeometric differential equation

$$0 = x(1-x)\frac{d^2F}{dx^2} + (\gamma - (\alpha+\beta+1)\,x)\,\frac{dF}{dx} - \alpha\beta F\,, \tag{8.24}$$

with the set of parameters (α,β,γ) being given by

$$\gamma = -\frac{1}{2}\,,\quad \alpha+\beta = \frac{1-3\lambda-\frac{3}{2}\mu}{3\mu}\,,\quad \alpha\beta = -\frac{1+3(\mu-\lambda)}{6\mu}\,. \tag{8.25}$$

The complete solution of Eq. (8.24) is a Gauss' hypergeometric function plus a linear term and a cosmological constant coming from the particular solution of Eq. (8.23), namely

$$F(\tilde{R}) = C_1F(\alpha,\beta,\gamma;x) + C_2x^{1-\gamma}F(\alpha-\gamma+1,\beta-\gamma+1,2-\gamma;x) + \frac{1}{\kappa_1}\tilde{R} - 2\Lambda\,. \tag{8.26}$$

where C_1 and C_2 are constants, $\kappa_1 = 3\lambda-1$ and $\Lambda = -\frac{3H_0^2(1-3\lambda+9\mu)}{2(1-3\lambda+3\mu)}$. Note that for the exact cosmology (8.21), the classical $F(R)$ gravity was reconstructed and studied in Chapter 5. In this case, the solution (8.26) behaves similarly to the classical $F(R)$ theory, except that now the parameters of the theory depend on (λ,μ), which are allowed to vary as it was noted above. One can also explore the solution (8.21) for a particular choice on the parameters $\mu = \lambda - \frac{1}{3}$, which plays a special role as it is shown below. In this case, the scalar $\tilde{R}$ turns out to be a constant, and Eq. (8.23), in the presence of a pressureless fluid, has the solution

$$F(\tilde{R}) = \frac{1}{\kappa_1}\tilde{R} - 2\Lambda\,,\quad \text{with}\quad \Lambda = \frac{3}{2}(3\lambda-1)H_0^2\,. \tag{8.27}$$

Hence, for this constraint on the parameters, the only consistent solution reduces to the Hořava linear theory with a cosmological constant.

As a further example, we consider the so-called phantom accelerating expansion. Currently, observational data do not totally exclude the possibility that the Universe could have already crossed the phantom

divide, which means that the effective EoS for dark energy would presently be slightly less than -1. Such kind of system can be easily expressed in GR, where the FRW equation reads $H^2 = \frac{\kappa^2}{3}\rho_{\rm ph}$. Here the subscript "ph" denotes the phantom nature of the fluid, which has an EoS given by $p_{\rm ph} = w_{\rm ph}\rho_{\rm ph}$ with $w_{\rm ph} < -1$. By using the energy conservation equation, the solution for the Hubble parameter turns out to be

$$H(t) = \frac{H_0}{t_s - t}\,, \tag{8.28}$$

where $H_0 = -1/3(1+w_{\rm ph})$, and t_s is the Rip time which represents the time still remaining up to the Big Rip singularity. As in the above example, one can rewrite the Hubble parameter as a function of the number of e-foldings; this yields

$$G(\eta) = H^2(\eta) = H_0^2 \mathrm{e}^{2\eta/H_0}\,. \tag{8.29}$$

Then, by using the expression of the scalar curvature, the relation between $\tilde{R}$ and η is given by

$$\mathrm{e}^{2\eta/H_0} = \frac{R}{H_0(AH_0 + 6\mu)}\,. \tag{8.30}$$

By inserting (8.29) and (8.30) into the differential equation (8.20), we get

$$\tilde{R}^2\frac{d^2F(\tilde{R})}{d\tilde{R}^2} + k_1\tilde{R}\frac{dF(\tilde{R})}{d\tilde{R}} + k_0F(\tilde{R}) = 0\,, \tag{8.31}$$

where

$$k_1 = -\frac{(AH_0+6\mu)((AH_0+3\mu))}{6\mu(AH_0+12\mu)}\,, \quad k_0 = \frac{(AH_0+6\mu)^2}{12\mu(AH_0+12\mu)}\,, \tag{8.32}$$

here we have neglected any kind of matter contribution for simplicity. Eq. (8.31) is an Euler equation, whose solution is well known

$$F(R) = C_1R^{m_+} + C_2R^{m_-}\,, \quad \text{where} \quad m_\pm = \frac{1-k_1 \pm \sqrt{(k_1-1)^2 - 4k_0}}{2}\,. \tag{8.33}$$

Hence, a $F(\tilde{R})$ HL gravity has been reconstructed that reproduces the phantom dark epoch with no need of any exotic fluid. In the same way, any given cosmology may be reconstructed.

8.3 Unified inflation and dark energy in modified Hořava-Lifshitz gravity

Let us consider here some viable $F(\tilde{R})$ gravities which admit the unification of inflation with late-time acceleration. In the convenient $F(R)$ theory, a number of viable models which pass all local tests and are able to unify the inflationary and the current cosmic accelerated epochs have been proposed, as it was commented in the preceeding chapters. Here we extend this class of models to the Hořava-Lifshitz gravity. We consider the action,

$$F(\tilde{R}) = \tilde{R} + f(\tilde{R})\,, \tag{8.34}$$

where it is assumed that the term $f(\tilde{R})$ becomes important at cosmological scales, while for scales compared with the Solar system one the theory becomes linear on $\tilde{R}$. As an example, we consider the following function, already analyzed in standard $F(R)$ gravity

$$f(\tilde{R}) = \frac{\tilde{R}^n(\alpha\tilde{R}^n - \beta)}{1+\gamma\tilde{R}^n}\,, \tag{8.35}$$

where (α, β, γ) are constants and $n > 1$. This theory reproduces the inflationary and cosmic acceleration epochs in convenient $F(R)$ gravity, which is also the case in the present theory, as will be shown. During inflation, it is assumed that the curvature scalar tends to infinity. In this case the model (8.34), with (8.35), behaves as

$$\lim_{\tilde{R}\to\infty} F(\tilde{R}) = \alpha \tilde{R}^n . \tag{8.36}$$

Then, by solving the FRW equation (8.15), this kind of function yields a power-law solution of the type

$$H(t) = \frac{h_1}{t}, \quad \text{where} \quad h_1 = \frac{2\mu(n-1)(2n-1)}{1 - 3\lambda + 6\mu - 2n(1 - 3\lambda + 3\mu)} . \tag{8.37}$$

This solution produces acceleration during the inflationary epoch if the parameters of the theory are properly defined. The acceleration parameter is given by $\frac{\ddot{a}}{a} = h_1(h_1 - 1)/t^2$, thus, for $h_1 > 1$ the inflationary epoch is well reproduced by the model (8.35). On the other hand, the function (8.35) has a minimum at $\tilde{R}_0$, given by

$$\tilde{R}_0 \sim \left(\frac{\beta}{\alpha\gamma}\right)^{1/4}, \qquad f'(\tilde{R}) = 0 , \qquad f(\tilde{R}) = -2\Lambda \sim -\frac{\beta}{\gamma} , \tag{8.38}$$

where we have imposed the condition $\beta\gamma/\alpha \gg 1$. Then, at the current epoch the scalar curvature acquires a small value which can be fixed to coincide with the minimum (8.38), such that the FRW equations (8.13) and (8.15) yield

$$H^2 = \frac{\kappa^2}{3(3\lambda - 1)}\rho_m + \frac{2\Lambda}{3(3\lambda - 1)} \qquad \dot{H} = -\kappa^2 \frac{\rho_m + p_m}{3\lambda - 1} , \tag{8.39}$$

which look very similar to the standard FRW equations in GR, except for the parameter λ. As has been pointed out, at the current epoch the scalar $\tilde{R}$ is small, so the theory is in the IR limit where the parameter $\lambda \sim 1$, and the equations approach the usual ones for $F(R)$ gravity. Hence, the FRW equations (8.39) reproduce the behavior of the well known Λ CDM model with no need to introduce a dark energy fluid to explain the current universe acceleration.

As another example of the models described by (8.34), we can considered the function,

$$f(\tilde{R}) = -\frac{(\tilde{R} - \tilde{R}_0)^{2n+1} + \tilde{R}_0^{2n+1}}{f_0 + f_1\left[(\tilde{R} - \tilde{R}_0)^{2n+1} + \tilde{R}_0^{2n+1}\right]} = -\frac{1}{f_1} + \frac{f_0/f_1}{f_0 + f_1\left[(\tilde{R} - \tilde{R}_0)^{2n+1} + \tilde{R}_0^{2n+1}\right]} . \tag{8.40}$$

This function could also serve for the unification of inflation and cosmic acceleration but, in this case, when one takes the limit $\tilde{R} \to \infty$, one gets

$$\lim_{\tilde{R}\to\infty} F(\tilde{R}) = \tilde{R} - 2\Lambda_i , \quad \text{where} \quad \Lambda_i = 1/2f_1 , \tag{8.41}$$

where the subscript i denotes that we are in the inflationary epoch. By inserting this into Eqs. (8.13) and (8.15), the FRW equations take the same form as in (8.39). Then, for the function (8.40) the inflationary epoch is produced by an effective cosmological constant, which implies that the parameter $\lambda > 1/3$, or the equations themselves will present inconsistencies, as it was discussed in the above section. For the current epoch, it is easy to see that the function (8.40) exhibits a minimum for $\tilde{R} = \tilde{R}_0$, which implies, as in the model above, an effective cosmological constant for late time that can produce the cosmic acceleration. The emergence of matter dominance before the dark energy epoch can be exhibited, in analogy with the case of the convenient theory. Hence, we have shown that the model (8.40) also unifies the cosmic expansion history, although with different properties during the inflationary epoch as compared with the model (8.35). This could be very important for the precise study of the evolution of the parameters of the theory.

It is also interesting to explore the de Sitter solutions allowed by the theory (8.7). By taking $H(t) = H_0$, the FRW equation (8.15), in absence of any kind of matter and with $C = 0$, reduces to

$$0 = F(\tilde{R}_0) - 6H_0^2(1 - 3\lambda + 3\mu)F'(\tilde{R}_0)\,, \tag{8.42}$$

which reduces to an algebraic equation that, for an specific model, can be solved yielding the possible de Sitter points allowed by the theory. As an example, let us consider the model (8.35), where Eq. (8.42) takes the form

$$\tilde{R}_0 + \frac{\tilde{R}_0^n(\alpha\tilde{R}_0^n - \beta)}{1+\gamma\tilde{R}_0^n} + \frac{6H_0^2(-1+3\lambda-3\mu)\left[1 + n\alpha\gamma\tilde{R}_0^{3n-1} + \tilde{R}_0^{n-1}(2\gamma\tilde{R}_0 - n\beta) + \tilde{R}_0^{2n-1}(\gamma^2\tilde{R}_0 + 2n\alpha)\right]}{(1+\gamma\tilde{R}_0^n)^2} = 0\,. \tag{8.43}$$

Here $\tilde{R}_0 = 3(1 - 3\lambda + 6\mu)H_0^2$. By specifying the free parameters of the theory, one can solve Eq. (8.43), which yields several de Sitter points, as the one studied above. They can be used to explain the coincidence problem, with the argument that the present will not be the only late-time accelerated epoch experienced by our Universe. In standard $F(R)$, it was found for this same model that it contains at least two de Sitter points along the cosmic history. In the same way, the second model studied here (8.40), provides several de Sitter points in the course of the cosmic history. Note that when $\mu = \lambda - \frac{1}{3}$, Eq. (8.42) turns out to be much more simple, it reduces to $F(\tilde{R}_0) = 0$, where the de Sitter points are the roots. For example, for (8.43) we have $\tilde{R}_0(1 + \gamma\tilde{R}_0^n) + \tilde{R}_0^n(\alpha\tilde{R}_0^n - \beta) = 0$, where the number of positive roots (de Sitter points) depends on the free parameters of the theory.

Summing up, it has been here shown that, also in $F(\tilde{R})$ Hořava-Lifshitz gravity, the so-called viable models, as (8.35) or (8.40), can in fact reproduce the whole cosmological history of the universe, with no need to involve any extra fields or a cosmological constant.

8.4 Newton law corrections in $F(\tilde{R})$ gravity

As is well-known, modified gravity may lead to violations of local tests. We explore in this section how to avoid these violations of Newton's law. It is known that $F(\tilde{R})$ theories include scalar particle. This scalar field could give rise to a fifth force and to variations of the Newton law, which can be avoided by a kind of the so-called chameleon mechanism [183].

In the original Hořava gravity, the projectability condition may cause problems with the Newton law [32] but the model without the projectability condition could be inconsistent for the $F(R)$-model [101]. The Newton law in the Hořava gravity may be restored by the "healthy" extension [32]. In this section, we do not discuss the gravity sector corresponding to the Hořava gravity but we show that the scalar mode, which also appears in the usual $F(R)$ gravity, can decouple from gravity and matter, and then the scalar mode does not give a measurable correction to Newton's law.

To show this, we consider a function of the type (8.34), and rewrite action (8.7) as

$$S = \int dtd^3x\sqrt{g^{(3)}}N\left[(1+f'(A))(\tilde{R} - A) + A + f(A)\right]\,, \tag{8.44}$$

where A is an auxiliary scalar field. It is easy to see that variation of action (8.44) over A gives $A = \tilde{R}$. Performing the conformal transformation $g_{ij}^{(3)} = \mathrm{e}^{-\phi}\tilde{g}_{ij}^{(3)}$, with $\phi = \frac{2}{3}\ln(1 + f'(A))$, action (8.44) yields

$$S = \int dtd^3x\sqrt{g^{(3)}}\left[\tilde{K}_{ij}\tilde{K}^{ij} - \lambda\tilde{K}^2 + \left(-\frac{1}{2} + \frac{3}{2}\lambda - \frac{3}{2}\mu\right)\dot{\tilde{g}}^{ij(3)}\tilde{g}_{ij}^{(3)}\dot{\phi} + \left(\frac{3}{4} - \frac{9}{4}\lambda + \frac{9}{2}\mu\right)\dot{\phi}^2 - V(\phi) + \tilde{L}(\tilde{g}^{(3)}, \phi)\right]\,, \tag{8.45}$$

where $\tilde{L}(\tilde{g}^{(3)}, \phi)$ is the conformally transformed term in (8.8), and the scalar potential is

$$V(\phi) = \frac{A(\phi)f'(A(\phi)) - f(A(\phi))}{1 + f'(A(\phi))} . \tag{8.46}$$

Note that, differently from the convenient $F(R)$, in action (8.45) there is a coupling term between the scalar field ϕ and the spatial metric $\tilde{g}_{ij}^{(3)}$, which can thus be dropped, by imposing the following condition on the parameters [90]:

$$\mu = \lambda - \frac{1}{3} . \tag{8.47}$$

This condition also renders the theory power-counting renormalizable, for the same z as in the original Hořava model.

Let us now investigate the term (8.8). As already pointed out, for local scales, where the scalar curvature is assumed to be very small, the theory enters the IR limit, where such term could be written as a spatial curvature,

$$L^{(3)}(g^{(3)}, \phi) \sim R^{(3)} . \tag{8.48}$$

Then, corrections to the Newton law will come from the coupling that now appears between the scalar field and matter, which makes a test particle to deviate from its geodesic path, unless the mass of the scalar field is large enough (since then the effect could be very small). The precise value can be calculated from

$$m_\phi^2 = \frac{1}{2}\frac{d^2V(\phi)}{d\phi^2} = \frac{1 + f'(A)}{f''(A)} - \frac{A + f(A)}{1 + f'(A)} . \tag{8.49}$$

In view of that, we can now analyze the models studied in the last section. We are interested to see the behavior at local scales, as on Earth, where the scalar curvature is around $A = \tilde{R} \sim 10^{-50}\mathrm{eV}^2$, or in the solar system, where $A = \tilde{R} \sim 10^{-61}\mathrm{eV}^2$. The function (8.35) and its derivatives can be approximated around these points as

$$f(\tilde{R}) \sim -\frac{\beta}{\gamma}, \quad f'(\tilde{R}) \sim \frac{n\alpha}{\gamma}R^{n-1}, \quad f''(\tilde{R}) \sim \frac{n(n-1)\alpha}{\gamma}R^{n-2} . \tag{8.50}$$

Then, the scalar mass for the model (8.35) is given approximately by the expression

$$m_\phi^2 \sim \frac{\gamma\tilde{R}^{2-n}}{n(n-1)\alpha} , \tag{8.51}$$

which becomes $m_\phi^2 \sim 10^{50n-100}\mathrm{eV}^2$ on Earth and $m_\phi^2 \sim 10^{61n-122}$ in the Solar System. We thus see that, for $n > 2$, the scalar mass would be sufficiently large in order to avoid corrections to the Newton law. Even for the limiting case $n = 2$, the parameters γ/α can be chosen to be large enough so that any violation of the local tests is avoided.

For the model (8.40), the situation is quite similar. For simplicity, we impose the following condition

$$f_0 \ll f_1 \left[(\tilde{R} - \tilde{R}_0)^{2n+1} + \tilde{R}_0^{2n+1}\right] \sim f_1\tilde{R}^{2n+1} . \tag{8.52}$$

Then, function (8.40) and its derivatives can be written, for small values of the curvature, as

$$f(\tilde{R}) \sim -\frac{1}{f_1} + \frac{f_0}{f_1^2 R^{2n+1}}, \quad f'(\tilde{R}) \sim -\frac{(2n+1)f_0}{f_1^2\tilde{R}^{2(n+1)}}, \quad f''(\tilde{R}) \sim \frac{2(2n+1)(2n+2)f_0}{f_1^2\tilde{R}^{2n+3}} . \tag{8.53}$$

Using now the expression for the scalar mass (8.49), this yields

$$m_\phi^2 \sim \frac{f_1^2\tilde{R}^{2n+3}}{2(2n+1)(n+1)f_0} + \frac{1}{f_1} . \tag{8.54}$$

Hence, as $\tilde{R}$ is very small at solar or Earth scales ($\sim 10^{-50\sim-61}\mathrm{eV}^2$), and $\Lambda_i = \frac{1}{2f_1} \sim 10^{20\sim38}\mathrm{eV}^2$ is the effective cosmological constant at inflation, which is much larger than the other term, it turns out that the second term on the rhs of (8.54) will dominate, and the scalar mass will take a value such as $m_\phi^2 \sim \frac{1}{f_1} \sim 10^{20\sim38}\mathrm{eV}^2$, which is large enough as compared with the scalar curvature. As a consequence there is no observable correction to Newton's law.

We have thus shown, in all detail, that the viable models (8.35) and (8.40) do not introduce any observable correction to the Newton law at small scales. This strongly supports the choice of $F(\tilde{R})$ gravity as a realistic candidate for the unified description of the cosmological history.

8.5 Finite-time future singularities in $F(\tilde{R})$ gravity

It is a well-known that a good number of effective phantom/quintessence-like dark energy models end their evolution at a finite-time future singularity. In the current section, we study the possible future evolution of the viable $F(R)$ Hořava-Lifshitz gravity considered above. It has been already proven [90] that power-law $F(R)$ HL gravities may lead, in its evolution, to a finite-time singularity. In order to properly define the type of future singularities, let us rewrite the FRW Eqs. (8.13) and (8.15) in the following way

$$3H^2 = \frac{\kappa^2}{3\lambda - 1}\rho_{\mathrm{eff}}\,, \quad -3H^2 - 2\dot{H} = \frac{\kappa^2}{3\lambda - 1}p_{\mathrm{eff}}\,, \tag{8.55}$$

where

$$\begin{aligned}
\rho_{\mathrm{eff}} &= \frac{1}{\kappa^2 F'(\tilde{R})}\left[-F(\tilde{R}) + 3(1-3\lambda+9\mu)H^2F'(\tilde{R}) - 6\mu\dot{H}F'(\tilde{R}) - 6\mu H\dot{\tilde{R}}F''(\tilde{R}) + \kappa^2\rho_m\right]\,,\\
p_{\mathrm{eff}} &= \frac{1}{\kappa^2 F'(\tilde{R})}\left[F(\tilde{R}) - (6\mu\dot{H} + 3(1-3\lambda+9\mu)H^2)F'(\tilde{R}) - 2(1-3\lambda)\dot{\tilde{R}}F''(\tilde{R})\right.\\
&\quad \left.+2\mu\left[\dot{\tilde{R}}^2F^{(3)}(\tilde{R}) + \ddot{\tilde{R}}F''(\tilde{R})\right] + \kappa^2 p_m\right]\,.
\end{aligned} \tag{8.56}$$

Then, using the expressions for the effective energy and pressure densities just defined, the list of future singularities, as classified in Ref. [246], can be extended to $F(\tilde{R})$ gravity as follows:

- Type I ("Big Rip"): For $t \to t_s$, $a \to \infty$ and $\rho_{\mathrm{eff}} \to \infty$, $|p| \to \infty$.
- Type II ("Sudden"): For $t \to t_s$, $a \to a_s$ and $\rho_{\mathrm{eff}} \to \rho_s$, $|p_{\mathrm{eff}}| \to \infty$.
- Type III: For $t \to t_s$, $a \to a_s$ and $\rho_{\mathrm{eff}} \to \infty$, $|p_{\mathrm{eff}}| \to \infty$.
- Type IV: For $t \to t_s$, $a \to a_s$ and $\rho_{\mathrm{eff}} \to \rho_s$, $p_{\mathrm{eff}} \to p_s$ but higher derivatives of Hubble parameter diverge.

To illustrate the possibility of future singularities in viable $F(\tilde{R})$ gravity, we explore the model (8.35), which has been studied in Ref. [239] for the convenient $F(R)$ case, where it was shown that this model is non-singular, in the particular case $n = 2$. Also, it is known that for most models of $F(R)$ gravity, the future singularity can be cured by adding a term proportional to R^2 (see Ref. [1]) or a non-singular modified gravity action [239]. However, this is not possible to do in the Hořava-Lifshitz gravity where, even for the simple model (8.35) with $n = 2$, a singularity can occur unless some restrictive conditions on the parameters are imposed.

In order to study the possible singularities that may occur from the above list, we consider a Hubble parameter close to one of such singularities, given by the following expression

$$H(t) = \frac{h_0}{(t_s - t)^q}\,, \tag{8.57}$$

where h_0 and q are constant. Depending on the value of q, the Hubble parameter (8.57) gives rise to a particular type of singularity. Thus, for $q \geq 1$, it gives a Big Rip singularity, for $-1 < q < 0$ a Sudden Singularity, for $0 < q < 1$ a Type III singularity and for $q < -1$, it will produce a Type IV singularity. Using the expression (8.12) with (8.57), the scalar curvature can be approximated, depending on the value of q, as

$$R \sim \begin{cases} \frac{3(1-3\lambda+6\mu)h_0^2}{(t_s-t)^{2q}} & \text{for} \quad q > 1 \\ \frac{3(1-3\lambda+6\mu)h_0^2+6\mu q h_0}{(t_s-t)^2} & \text{for} \quad q = 1 \\ \frac{6\mu q h_0}{(t_s-t)^{q+1}} & \text{for} \quad q < 1 \end{cases} . \tag{8.58}$$

The model (8.35) which, as has been shown, unifies the inflationary and the dark energy epochs, makes the scalar curvature grow with time so that, close to a possible future singularity, the model can be approximated as $F(\tilde{R}) \sim \tilde{R}^n$, where $n > 1$. For the case $q > 1$, it is possible to show, from the FRW Eqs. (8.13) and (8.15), that the solution (8.57) is allowed for this model just in some special cases: (i) For $n = 3/2$ and $\mu = 2\lambda - \frac{2}{3}$, which contradicts the decoupling condition (8.47) and the Newtonian corrections (where it was found that $n > 2$). (ii) When $\mu = 0$ and $\lambda = 1/3$, which holds (8.47) and fixes the values of the parameters, in contradiction with the fact that fluctuations are allowed for them.

For $q = 1$, the Hubble parameter (8.57) is a natural solution for this model, yielding

$$H(t) = \frac{h_0}{t_s - t} \quad \text{with} \quad h_0 = \frac{2\mu(n-1)(2n-1)}{-1+3\lambda-6\mu+2n(1-3\lambda+3\mu)}\,. \tag{8.59}$$

This model allows for the possibility of occurrence of a Big Rip singularity, unless the parameters of the theory are fixed. As pointed out in Ref. [239], for the case $n = 2$ the Big Rip singularity can be avoided in standard $F(R)$ gravity, which can be easily seen by choosing $\lambda = \mu = 1$ in the solution (8.59). Nevertheless, in $F(\tilde{R})$ gravity, in order to avoid such singularity, the power n of the model has to be fixed to the value

$$n = \frac{1}{2} + \frac{3\mu}{2 - 6\lambda + 6\mu}\,, \tag{8.60}$$

which can be interpreted as another constraint on the parameters of the theory (although, when the decoupling condition (8.47) is satisfied, no constraints can be imposed on n). In addition, note that $h_0 = -h_1$ in (8.37), where we imposed $h_1 > 1$ with the aim to reproduce the inflationary epoch, so that h_0 would be negative and the solution (8.59) would correspond to the Big Bang singularity. Thus, no future doomsday will take place.

For the last case, when $q < 1$, we find that the Hubble parameter (8.57) can be a consistent solution when Eq. (8.47) is satisfied and $q = (n(1-n)-1)/n(n-2)$, although if we impose $n \geq 2$, as it was found above, $q < -1$, which implies a future singularity of the type IV. The exception here is when $n = 2$, that avoids the occurrence of any type of singularity when $q < 1$. Nevertheless, even in the case that, for any reason, some kind of future singularity would be allowed in the model (8.35), one must also take into account possible quantum gravity effects, which may probably become important when one is close to the singularity. They have been shown, in phantom models, to prevent quite naturally the occurrence of the future singularity [133].

In summary, we have here proven that the theory (8.35) can actually be free of future singularities, and thus that it can make a good candidate for the unification of the cosmic history. Note that, from

the corrections to the Newton law studied in the previous section, as well as from imposing avoidance of future singularities, we can fix some of the parameters of the theory. The other constant parameters that appear in (8.35)—which expresses the algebraic relation between the powers of the scalar curvature—could be fixed by comparing the cosmic evolution with the observed data, quite in the same way as has been successfully done for standard $F(R)$ gravity in Chapter 6.

8.6 Discussion

In summary, we have here investigated the FRW cosmology of a non-linear modified Hořava-Lifshitz $F(R)$ gravity theory which has a viable convenient counterpart. We have proven that, for a special choice of the parameters, the FRW equations are just the same in both theories, and that the cosmic history of the first literally coincides with the one for the viable $F(R)$ gravity. For a more general version of the theory, the unified description of the early-time inflation and late-time acceleration is proven to be possible too; however, the details of the cosmological dynamics are here different. Moreover, corrections to Newton's law are negligible for an extensive region of the parameter space. We have demonstrated the emergence of possible finite-time future singularities, and their avoidance, by adding extra higher-derivative terms which turn out to be qualitatively different, as compare with conventional $F(R)$ cosmology.

Using the approach shown in Chapter 5, one can construct more complicated generalizations of modified gravity with anisotropic scaling properties. For instance, one can obtain non-local or modified Gauss-Bonnet Hořava-Lifshitz gravities. Technically, this is of course a more involved task, as compared with the $F(R)$ case. The corresponding cosmology can in principle be studied in the same way as in the present chapter, with expected qualitatively similar results, owing to the fact that the convenient cosmological model has been already investigated, in those cases. It is foreseen that the study of such theories will help us also in the resolution of the some problems for the theories with broken Lorentz symmetry. For instance, it is quite possible that a natural scenario of dynamical Lorentz symmetry breaking, which would reduce gravity to the Hořava limit, may be found in this way, what would be certainly interesting.

Chapter 9

Stability of cosmological solutions in F(R) Hořava-Lifshitz gravity

[1]In the previous chapter, we showed how modifications of the original Hořava-Lifshitz action can unify well inflation and late-time acceleration providing a natural explanation on both effects. In the present chapter, general cosmological solutions of the type of spatially flat FLRW will be studied in the frame of F(R) Hořava-Lifshitz gravity, and its stability is explored. We specially focus on the study of stability of radiation/matter dominated eras, where the Universe expands following a power law, and de Sitter solutions, which can well shape the accelerated epochs of Universe evolution. We explore space independent perturbations around these solutions, studying the effects of the extra terms introduced in the field equations by considering $F(\tilde{R})$ gravity. Also the new parameters included in the theory, due to the breaking of the Lorentz invariance, could affect to the Universe evolution. The (in)stability of a cosmological solution gives very important information as the possible exit from one phase of the cosmological history, constraints on the kind of action $F(R)$ and future observational predictions. An explicit example of a $F(\tilde{R})$ action, where an unstable de Sitter solution is found, is studied. This model performs a successful exit from inflation, and produces an instability at the end of matter dominated epoch, such that a phase transition may occur.

9.1 Cosmological solutions and its stability in $F(\tilde{R})$ gravity

In this section, we are interested to study the stability of general cosmological solutions in the frame of $F(\tilde{R})$ Hořava-Lifshitz gravity, with special attention to those cosmological solutions that model the history of the Universe, as de Sitter or power law solutions. As it was shown in the previous chapter, dark energy and even the unification with the inflationary epoch can be reproduced in this new frame of $F(\tilde{R})$ Hořava-Lifshitz theories. The stability of those solutions plays a crucial role in order to get the transition from one cosmological phase to another.

[1]This Chapter is based on: [269]

9.1.1 Stability of general flat FLRW cosmological solutions

Let us start by studying a general spatially flat FLRW metric. We will focus below specially on de Sitter and power law solutions of the type $a(t) \propto t^m$ as dark energy epoch and the radiation/matter dominated eras are governed by this class of cosmological solutions respectively, so that the implications of the extra geometrical terms coming from $F(\tilde{R})$ could be determinant for the stability and transition during those epochs. We assume a general solution,

$$H(t) = h(t) \ . \tag{9.1}$$

Then, the scalar curvature $\tilde{R}$ yields,

$$\tilde{R}_h(t) = 3(1 - 3\lambda + 6\mu)h^2(t) + 6\mu\dot{h}(t) \ . \tag{9.2}$$

Assuming that solution (9.1) is satisfied by a particular choice of $F(\tilde{R})$, the FLRW equation has to be fulfilled,

$$0 = F(\tilde{R}_h) - 6\left[(1 - 3\lambda + 3\mu)h^2 + \mu\dot{h}\right] F'(\tilde{R}_h) + 6\mu h\dot{\tilde{R}}_h F''(\tilde{R}_h) - \kappa^2\rho_m \, , \tag{9.3}$$

where the matter fluid is taken to be a perfect fluid with equation of state $p_m = w_m\rho_m$, with w_m constant. By the energy conservation equation $\dot{\rho}_m + 3h(1 + w_m)\rho_m = 0$, the evolution of the matter energy density can be expressed in terms of the given solution $h(t)$ as,

$$\rho_{mh} = \rho_0 \mathrm{e}^{-3(1+w_m)\int h(t)dt} \ . \tag{9.4}$$

being ρ_0 an integration constant. We are interested to study the perturbations around the solution $h(t)$. For that purpose, let us expand the function $F(\tilde{R})$ in powers of $\tilde{R}$ around (9.2),

$$F(\tilde{R}) = F_h + F_h'(\tilde{R} - \tilde{R}_h) + \frac{F_h''}{2}(\tilde{R} - \tilde{R}_h)^2 + \frac{F_h^{(3)}}{6}(\tilde{R} - \tilde{R}_h)^3 + O(\tilde{R} - \tilde{R}_h)^4 \ , \tag{9.5}$$

where the derivatives of the function $F(\tilde{R})$ are evaluated at R_h, given in (9.2). Note that matter perturbations also contribute to the stability, inducing a mode on the perturbation. Then, we can write the perturbed solution as,

$$H(t) = h(t) + \delta(t) \ , \quad \rho_m \simeq \rho_{mh}(1 + \delta_m(t)) \ . \tag{9.6}$$

Hence, by introducing the above quantities in the FLRW equation, the equation for the perturbation $\delta(t)$ becomes (in the linear approximation),

$$\ddot{\delta} + b\dot{\delta} + \omega^2\delta = \frac{\kappa^2\rho_{mh}}{36\mu^2 h F_h''}\delta_m \ , \tag{9.7}$$

where,

$$b = -\frac{h'}{h} - \frac{1 - 3\lambda + 3\mu}{\mu}h + \frac{1 - 3\lambda + 6\mu}{\mu} + 6\left((1 - 3\lambda + 6\mu)h\dot{h} + \mu\ddot{h}\right)\frac{F_h^{(3)}}{F_h''} \ ,$$

$$\omega^2 = \left[1 - 3\lambda + 6\mu - 2(1 - 3\lambda + 3\mu)h\right]\frac{F_h'}{6\mu^2 h F_h''}$$

$$+ (1 - 3\lambda + 6\mu)\left(\frac{-1 + 3\lambda - 3\mu}{\mu^2}h + \frac{-1 + h}{\mu h}\dot{h}\right) + \frac{\ddot{h}}{h} + 6(1 - 3\lambda + 6\mu)\left(\frac{1 - 3\lambda + 6\mu}{\mu}\dot{h}h + \ddot{h}\right)\frac{F_h^{(3)}}{F_h''} \ . \tag{9.8}$$

In this case, the solution for $\delta(t)$ can be split in two branches, one corresponding to the homogeneous part of the equation (9.7), whose solution will depend on the background theory, i.e. on $F(\tilde{R})$ and its

derivatives, and the other one corresponding to the particular solution of the eq. (9.7), which represent the solution induced by the matter perturbation, δ_m . Then, the complete solution can be written as,

$$\delta(t) = \delta_{homg}(t) + \delta_{inh}(t) \ . \tag{9.9}$$

We are interested on the perturbations induced by the function $F(\tilde{R})$ and its derivatives, so that we focus on the homogeneous solution, δ_{homg} . By a first qualitative analysis, we can see that the homogeneous part of equation (9.7) yields exponential or damped oscillating perturbations. The form of the perturbations will depend completely on the form of the function $F(\tilde{R})$ and its derivatives evaluated at R_h . Note that in general, the equation (9.7) has to be solved by numerical methods. Nevertheless, we could assume some restrictions to obtain qualitative information. Let us consider the cases,

- The trivial case, given by $F'_h = F''_h = F^{(3)}_h = 0$, makes the perturbation tends to zero, $\delta(t) = 0$, and the cosmological solution $h(t)$ is stable.
- For $F'_h \neq 0$ and $F''_h, F^{(3)}_h \to 0$, the term that dominates in (9.7) is given by

$$\omega^2 \sim [1 - 3\lambda + 6\mu - 2(1 - 3\lambda + 3\mu)h] \frac{F'_h}{6\mu^2 h F''_h} \ . \tag{9.10}$$

  And the stability of the cosmological solution $h(t)$ depends on the sign of this term, and therefore, on the model $F(\tilde{R})$ and the solution $h(t)$.
- For $F'_h, F''_h \to 0$ but $F^{(3)}_h \neq 0$, looking at (9.8), the perturbation depends on the value of the last term in the coefficients b and ω^2 , which can be approximated to,

$$b \sim 6\left((1 - 3\lambda + 6\mu)h\dot{h} + \mu\ddot{h}\right) \frac{F^{(3)}_h}{F''_h} \ , \quad \omega^2 \sim 6(1 - 3\lambda + 6\mu)\left(\frac{1 - 3\lambda + 6\mu}{\mu}\dot{h}h + \ddot{h}\right) \frac{F^{(3)}_h}{F''_h} \ . \tag{9.11}$$

  The cosmological solution will be stable in the case that both coefficients (9.11) are greater than zero, what yields a damped oscillating perturbation that decays.

However, in general the equation (9.7) can not be solved analytically for arbitrary solutions $h(t)$ and actions $F(\tilde{R})$, and numerical analysis is required. Nevertheless, by imposing certain conditions on $F(\tilde{R})$ as above, qualitative information can be obtained. In order to perform a deeper analysis, some specific solutions $h(t)$ are studied below, as well as an explicit example of $F(\tilde{R})$.

9.1.2 de Sitter solutions in $F(\tilde{R})$ gravity

Let us consider one of the simplest but most important solution in cosmology, de Sitter (dS) solution. As dark energy and inflation can be shaped (in its simplest form) by a dS solution, its stability becomes very important, specially in the case of inflation, where a successful exit is needed to end the accelerated phase occurred during the early Universe. In general, standard $F(R)$ gravity contains several de Sitter points, which represent critical points (see [109]). The analysis can be extended to $F(\tilde{R})$ Hořava-Lifshitz gravity, where the de Sitter solution $H(t) = H_0$, with H_0 being a constant, has to satisfy the first equation FLRW equation,

$$0 = F(\tilde{R}_0) - 6H_0^2(1 - 3\lambda + 3\mu)F'(\tilde{R}_0) \ , \tag{9.12}$$

where we have taken $C = 0$ and assumed absence of any kind of matter. The scalar $\tilde{R}$ is given in this case by,

$$\tilde{R}_0 = 3(1 - 3\lambda + 6\mu)H_0^2 \ . \tag{9.13}$$

Then, the positive roots of equation (9.12) are the de Sitter points allowed by a particular choice of $F(\tilde{R})$. By assuming a de Sitter solution, we expand $F(\tilde{R})$ as a series of powers of the scalar $\tilde{R}$ around $\tilde{R}_0$,

$$F(\tilde{R}) = F_0 + F_0'(\tilde{R} - \tilde{R}_0) + \frac{F_0''}{2}(\tilde{R} - \tilde{R}_0)^2 + \frac{F_0^{(3)}}{6}(\tilde{R} - \tilde{R}_0)^3 + O(\tilde{R}^4) \ . \tag{9.14}$$

Here, the primes denote derivative respect $\tilde{R}$ while the subscript 0 means that the function $F(\tilde{R})$ and its derivatives are evaluated at $\tilde{R}_0$. Then, by perturbing the solution, the Hubble parameter can be writing as,

$$H(t) = H_0 + \delta(t) \ . \tag{9.15}$$

Using (9.14), and the perturbed solution (9.15) in the first FLRW equation, the equation for the perturbation yields,

$$\begin{aligned} 0 = \frac{1}{2}F_0 - 3H_0^2(1 - 3\lambda + 3\mu) - 3H_0 \left[\left((1 - 3\lambda)F_0' + 6F_0'' H_0^2(-1 + 3\lambda - 6\mu)(-1 + 3\lambda - 3\mu) \right) \delta(t) \right. \\ \left. + 6F_0''\mu H_0(-1 + 3\lambda - 3\mu)\dot{\delta}(t) - 12F_0''\mu^2\ddot{\delta}(t) \right] \ . \end{aligned} \tag{9.16}$$

Here we have restricted the analysis to the linear approximation on δ and its derivatives. Note that the first two terms in the equation (9.16) can be removed because of equation (9.12), which is assumed to be satisfied, and equation (9.16) can be rewritten in a more convenient form as,

$$\ddot{\delta}(t) + \frac{H_0(1 - 3\lambda + 9\mu)}{2\mu}\dot{\delta}(t) + \frac{1}{12\mu^2}\left[(3\lambda - 1)\frac{F_0'}{F_0''} - 6H_0^2(1 - 3\lambda + 6\mu)(1 - 3\lambda + 3\mu) \right] \delta(t) = 0 \ . \tag{9.17}$$

Then, the perturbations on a dS solution will depend completely on the model, specifically on the derivatives of $F(\tilde{R})$, as well as on the parameters (λ, μ). The instability becomes large if the term in front of $\delta(t)$ (the frequency) in the equation (9.17) becomes negative and the perturbation grows exponentially, while if we have a positive frequency, the perturbation behaves as a damped harmonic oscillator. During dark energy epoch, when the scalar curvature is very small, the IR limit of the theory can be assumed, where $\lambda = \mu \sim 1$, and the frequency depends completely on the value of $\frac{F_0'}{F_0''}$. In order to avoid large instabilities during the dark energy phase, the condition $\frac{F_0'}{F_0''} > 12H_0^2$ has to be fulfilled. Nevertheless, when the scalar curvature is large, the IR limit is not a convenient approach, and the perturbation depends also on the values of (λ, μ). If we assume a very small F_0'', the frequency in the equation (9.17) dominates compared to the other terms, and by assuming $\lambda > 1/3$, the stability of the solution will depend on the sign of $\frac{F_0'}{F_0''}$, being stable when it is positive.

9.1.3 Stability of radiation/matter eras: Power law solutions

In this section, an important class of cosmological solutions is considered, the power law solutions, which are described by the Hubble parameter,

$$H(t) = \frac{m}{t} \quad \rightarrow \quad a(t) \propto t^m \ . \tag{9.18}$$

In the context of General Relativity, this class of solutions are generated by a perfect fluid with equation of state parameter $w = -1 + \frac{2}{3m}$, and the matter/radiation dominated epochs are approximately described by (9.18). Also phantom epochs can be described by this class of solutions when $m < 0$. Let us study the stability for the Hubble parameter (9.18), and how the inclusion of extra terms in the action and the new parameters (λ, μ) may affect the stability of the solution (9.18). As in the above section, the perturbation

equation (9.7) can not be solved analytically in general, although under some restrictions we can obtain important qualitative information about the stability of the solution. Then, by assuming an $F(\tilde{R})$ that approximately does not deviate from Hilbert-Einstein action during radiation/matter dominated epochs, the second and third derivatives can be neglected $F''_h, F^{(3)}_h \sim 0$ (as they must become important only during dark energy epoch and/or inflation). In such a case, the coefficient in front of $\delta(t)$ in the eq. (9.7) is approximated as,

$$\omega^2 \sim [1 - 3\lambda + 6\mu - 2(1 - 3\lambda + 3\mu)h(t)]\,\frac{F'_h}{6\mu^2 h(t) F''_h}\ . \tag{9.19}$$

Then, the value of the frequency ω^2 depends on the time, such that the stability may change along the phase. For small values of t, the frequency takes the form $\omega^2 \sim -2(1 - 3\lambda + 3\mu)\frac{6\mu^2 F'_h}{F''_h}$, and assuming $\lambda \sim \mu$, the perturbations will grow exponentially when $\frac{F'_h}{F''_h} > 0$, and the solution becomes unstable. While for large t, the frequency can be approximated as $\omega^2 \sim (1 - 3\lambda + 6\mu)\frac{F'_h}{6\mu^2 h(t) F''_h}$ and the instability will be large if $\frac{F'_h}{F''_h} < 0$, and a phase transition may occur.

9.2 Example of a viable $F(\tilde{R})$ model

Let us consider an explicit model of $F(\tilde{R})$ gravity in order to apply the analysis about the stability performed above. We are interested to study the stability of radiation/matter dominated eras as well as de Sitter solutions. The model considered here, a class of the so-called viable models, was proposed in Ref. [237], and a variant was studied in Chapter 6 in the context of standard gravity and generalized to Hořava-Lifshitz gravity in Chapter 9. Let us write the action

$$F(\tilde{R}) = \chi\tilde{R} + \frac{\tilde{R}^n(\alpha\tilde{R}^n - \beta)}{1 + \gamma\tilde{R}^n}\ , \tag{9.20}$$

where ($\chi, \alpha, \beta, \gamma$) is a set of constant parameters of the theory. In standard gravity, this model can reproduce well late-time acceleration with no need of a cosmological constant or any kind of exotic field, as well as also inflation, such that the unification of both epochs of the Universe history under the same mechanism can be performed (see Ref. [237]). For simplicity, we assume $n = 2$ in (9.20) for our analysis. The radiation/matter dominated epochs, which can be described by the class of solutions given in (9.18), could suffer a phase transition to the era of dark energy due to the instabilities caused by the second term of the action (9.20). Then, we are interested to study the possible effects produced by the presence of these extra geometric terms during the cosmological evolution. By assuming the solution (9.18), and following the steps described in the above section, the stability is affected by the derivatives of the function (9.20) evaluated at $h(t) = m/t$. We are interesting in large times, when the end of matter dominated epoch has to occur. At that moment the derivatives of $F(\tilde{R})$ can be approximated as,

$$F'_h \to \chi\ , \quad F''_h \to -2\beta\ , \quad F^{(3)}_h \to 0\ . \tag{9.21}$$

Here for simplicity, we have assumed $0 < \beta << 1$. By means of the analysis performed in the previous section, we can conclude that the linear perturbation $\delta(t)$ grows exponentially, and the radiation/matter dominated phase becomes unstable for large times, what may produce the transition to another different phase. Then, the $F(\tilde{R})$ function (9.20) can explain perfectly the end of matter dominated epoch with no need of the presence of a cosmological constant.

Let us now study the stability of de Sitter solutions. It is known that the above model (9.20) may contain several de Sitter solutions, which are the solutions of the algebraic equation (9.12),

$$\tilde{R}_0+\frac{\tilde{R}_0^n(\alpha\tilde{R}_0^n-\beta)}{1+\gamma\tilde{R}_0^n}+\frac{6H_0^2(-1+3\lambda-3\mu)\left[1+n\alpha\gamma\tilde{R}_0^{3n-1}+\tilde{R}_0^{n-1}(2\gamma\tilde{R}_0-n\beta)+\tilde{R}_0^{2n-1}(\gamma^2\tilde{R}_0+2n\alpha)\right]}{(1+\gamma\tilde{R}_0^n)^2}=0\,. \tag{9.22}$$

This equation has to be solved numerically, even for the simple case studied here $n=2$. Nevertheless, one of the de Sitter points of this model is defined by a minimum of the second term in the action (9.20). By assuming the constraint on the parameters $\beta\gamma/\alpha\gg 1$, the minimum that represents a de Sitter point is given by,

$$\tilde{R}_0\sim\left(\frac{\beta}{\alpha\gamma}\right)^{1/4}\,,\qquad F'(\tilde{R}_0)=\chi\,,\qquad F(\tilde{R}_0)=\tilde{R}_0-2\Lambda\,,\quad \text{where}\quad \Lambda\sim\frac{\beta}{2\gamma}\,. \tag{9.23}$$

Then, by evaluating the derivatives of (9.20) around $\tilde{R}_0$ and by the equation (9.17) the perturbation δ can be calculated. Note that the stability condition for de Sitter solution, given by $\frac{F_0'}{F_0''}>12H_0^2$, is not satisfied for this case as $F_0''>>F_0'$, such that the de Sitter point (9.23) is unstable. By solving eq. (9.17), the perturbation is given by exponential functions,

$$\delta(t)=C_1\mathrm{e}^{a_+t}+C_2\mathrm{e}^{a_-t}\,,\quad \text{with}\quad a_\pm=\frac{H_0(1-3\lambda+3\mu)}{2\mu}\,. \tag{9.24}$$

Hence, the model (9.20) is unstable around the de Sitter point (9.23), what predicts the exit from an accelerated phase in the near future, providing a natural explanation about the end of the inflationary epoch, or a future prediction about the end of dark energy era. However, the theory described by (9.20) may contain more de Sitter points, given by the roots of equation (9.22), which may be stable. Then, a deeper analysis has to be performed to study the entire Universe evolution for this model of $F(\tilde{R})$ gravity.

9.3 Discussions

At the present chapter, we have analyzed spatially flat FLRW cosmology for nonlinear Hořava-Lifshitz gravity. Basically we have extended standard $F(R)$ gravity to Hořava-Lifshitz theory, which reduces to the first one in the IR limit (where we assume that the parameters (λ,μ) are reduced to unity). The stability of this general class of solutions has been studied and it is shown that it depends mainly on the choice of the function $F(\tilde{R})$ and in part on the values of the parameters (λ,μ). For large times, when the scalar curvature is very small, the main effect of the perturbation on a cosmological solution is caused by the value of the derivatives of $F(\tilde{R})$. It is shown that in general, the perturbation equation can not be solved analytically, even in the linear approach. Nevertheless, under some restrictions, important information is obtained, and the (in)stability of the different phases of the Universe history can be studied. For specific values of the derivatives of $F(\tilde{R})$, a given solution can becomes (un)stable, which means a major constraint on models. By analyzing an explicit example in Sect. IV, where an $F(\tilde{R})$ function of the class of viable models is considered, we have found that this kind of theories can well explain the end of matter dominated epoch, and reproduces late-time acceleration. We have shown that for this specific example, there is a de Sitter point that becomes unstable, what predicts the end of this de Sitter epoch, providing a natural explanation of the end of inflationary era. However, as this model (and in general every $F(\tilde{R})$ model) may contain several de Sitter solutions, where some of them can be stable, a further analysis of the phase space has to be performed to connect the different regions of the Universe history.
Hence, the analysis made here provides a general approach for the study of spatially flat FLRW solutions in

the frame of higher order Hořava-Lifshitz gravities, which can restrict the class of functions $F(\tilde{R})$ allowed by the observations, and it gives a natural explanation of the end of inflation and matter dominated epoch, shaping the Universe history in a natural way.

Chapter 10

U(1) Invariant F(R) Hořava-Lifshitz Gravity

[1]The HL gravity is based on an idea that the Lorentz symmetry is restored in IR limit of given theory and can be absent at high energy regime of given theory. Explicitly, Hořava considered systems whose scaling at short distances exhibits a strong anisotropy between space and time, In $(D+1)$ dimensional space-time in order to have power counting renormalizable theory requires that $z \geq D$. It turns out however that the symmetry group of given theory is reduced from the full diffeomorphism invariance of General Relativity to the foliation preserving diffeomorphism The common property of all modified theories of gravity is that whenever the group of symmetries is restricted (as in Hořava-Lifshitz gravity) one more degree of freedom appears that is a spin-0 graviton. An existence of this mode could be dangerous for all these theories (for review, see [262]). For example, in order to have the theory compatible with observations one has to demand that this scalar mode decouples in the IR regime. Unfortunately, it seems that this might not be the case. It was shown that the spin-0 mode is not stable in the original version of the HL theory [174] as well as in SVW generalization [282]. Note that in both of these two versions, it was all assumed the projectability condition that means that the lapse function N depends on t only. This assumption has a fundamental consequence for the formulation of the theory since there is no local form of the Hamiltonian constraint but the only global one. Even if these instabilities indicate to problems with the projectable version of the HL theory it turns out that this is not the end of the whole story. Explicitly, these instabilities are all found around the Minkowski background. Recently, it was indicated that the de Sitter space-time is stable in the SVW setup [177] and hence it seems to be reasonable to consider de Sitter background as the natural vacuum of projectable version of the HL gravity. This may be especially important for the theories with unstable flat space solution.

On the other hand there is the second version of the HL gravity where the projectability condition is not imposed so that $N = N(\mathbf{x}, t)$. Properties of such theory were extensively studied in [27, 31, 32, 172, 192, 199, 259]. It was shown recently in [33] that so called healthy extended version of such theory could really be an interesting candidate for the quantum theory of reality without ghosts and without strong coupling problem despite its unusual Hamiltonian structure [190, 189]. Nevertheless, such theory is not free from its own internal problems.

Recently Hořava and Melby-Thompson [175] proposed very interesting way to eliminate the spin-0 graviton. They considered the projectable version of the HL gravity together with extension of the foliation

[1]This Chapter is based on: [191]

preserving diffeomorphism to include a local $U(1)$ symmetry. The resulting theory is then called as non-relativistic covariant theory of gravity. It was argued there [175] that the presence of this new symmetry forces the coupling constant λ to be equal to one. However, this result was questioned in [113] where an alternative formulation of non-relativistic general covariant theory of gravity was presented. Furthermore, it was shown that the presence of this new symmetry implies that the spin-0 graviton becomes non-propagating and the spectrum of the linear fluctuations around the background solution coincides with the fluctuation spectrum of General Relativity. This construction was also extended to the case of RFDiff invariant HL gravities [33, 187] in [186] where it was shown that the number of physical degrees of freedom coincides with the number of physical degrees of freedom in General Relativity.

The goal of this chapter is to extend above construction to the case of $F(\tilde{R})$ HL gravities. $F(\tilde{R})$ HL gravity can be considered as natural generalization of covariant $F(R)$ gravity as it was pointed out in the above chapters, where current interest to $F(R)$ gravity is caused by several important reasons. Additionally, it is expected that such modification may be helpful for resolution of internal inconsistency problems of the HL theory. Indeed, we will present the example of $F(\tilde{R})$ HL gravity which has stable de Sitter solution but unstable flat space solution. In such a case, the original scalar graviton problem formally disappears because flat space is not vacuum state. Hence, there is no sense to study propagators structure around flat space. The complete propagators structure should be investigated around de Sitter solution which seems to be the candidate for vacuum space.

10.1 On flat space solutions in $F(\tilde{R})$ gravity

Let us now study flat space solutions in $F(\tilde{R})$ HL gravity. In this section we restrict to the case of $3+1$ dimensional space-time. A general metric in the ADM decomposition in a $3+1$ space-time is given by,

$$ds^2 = -N^2dt^2 + g_{ij}^{(3)}(dx^i + N^i dt)(dx^j + N^j dt)\,, \tag{10.1}$$

where $i,j = 1,2,3$, N is the so-called lapse variable, and N^i is the shift 3-vector. For flat space the variables from the metric (10.1) take the values,

$$N = 1\,, \quad N_i = 0 \quad \text{and} \quad g_{ij} = g_0\delta_{ij}\,, \tag{10.2}$$

where g_0 is a constant. Then, the scalar curvature $\tilde{R} = 0$, and so that our theory has flat space solution, the function $F(\tilde{R})$ has to satisfy,

$$F(0) = 0\,. \tag{10.3}$$

Hence, we assume the condition (10.3) is satisfied, otherwise the theory has no flat space solution. We are interested to study the stability of such solutions for a general $F(\tilde{R})$, by perturbing the metric in vacuum (10.2), this yields,

$$N = 1 + \delta_N(t) \quad \text{and} \quad g_{ij} = g_0(1+\delta_g(t))\delta_{ij}\,. \tag{10.4}$$

For simplicity, we restrict the study on the time-dependent perturbations (no spatial ones) and on the diagonal terms of the metric. Note that, as we are assuming the projectability condition, by performing a transformation of the time coordinate, we can always rewrite $N=1$. Then, the study of the perturbations is focused on the spatial components of the metric g_{ij}, which can be written in a more convenient way as,

$$g_{ij} = g_0\left(1 + \int \delta(t)\right)\delta_{ij} \sim g_0 \mathrm{e}^{\int \delta(t)dt}\delta_{ij}\,. \tag{10.5}$$

By inserting (10.5) in the first field equation, obtained by the variation of the action on N, it yields at lowest order on $\delta(t)$,

$$12\mu^2 F_0''\ddot{\delta}(t)\delta(t) - 6\mu^2 F_0''\dot{\delta}^2(t) + (-1+3\lambda)F_0'\delta^2(t) = 0\,. \tag{10.6}$$

Here the derivatives F_0' , F_0'' are evaluated on $\tilde{R}=0$. Then, the perturbation δ will depend completely on the kind of theory assumed. We can study some general cases by imposing conditions on the derivatives of $F(\tilde{R})$.

- For the case $F_0''=0$, it gives $\delta(t)=0$, such that at lowest order the flat space is completely stable for this case.
- For $F_0'=0$ and $F_0''\neq 0$, the differential equation (10.6) has the solution,

$$\delta(t)=C_1 t\left(1+\frac{C_1 t}{C_2}\right)+C_2 \tag{10.7}$$

  where $C_{1,2}$ are integration constants. Then, for this case, the perturbations grow as the power of the time coordinate, and flat space becomes unstable.

Hence, depending on the theory, the flat space solution will be stable or unstable, which becomes very important as it could be used to distinguish the theories or analyze their consistency.

10.1.1 A simple example

Let us now discuss a simple example. We consider the function,

$$F(\tilde{R})=\kappa_0\tilde{R}+\kappa_1\tilde{R}^n\,, \tag{10.8}$$

where $\kappa_{0,1}$ are coupling constants and $n>1$. Note that this family of theories satisfies the condition (10.3). The values of first and second derivatives evaluated in the solution depend on the value of n in (10.8),

$$F'(0)=\kappa_0\,,\quad F''(0)=\kappa_1 n(n-1)\tilde{R}^{n-2}\,. \tag{10.9}$$

Then, we can distinguish between the cases,

- For $n\neq 2$, we have $F''(0)=0$, and by the analysis performed above, it follows that flat space is stable
- For $n=2$, we have $F''(0)=\kappa_1$, and flat space is unstable.

Hence, we have shown that the stability of solution depends completely on the details of the theory.

We can now analyze the de Sitter solution. Using the the analysis performed in the above chapter, we can find the de Sitter points allowed by the class of theories given in Eq. (10.8),

$$\frac{3}{2}H_0^2(3\lambda-1)\kappa_0+\frac{\kappa_1\left(3H_0^2(1-3\lambda+6\mu)\right)^n(1-3\lambda+6\mu-2n(1-3\lambda+3\mu))}{2(1-3\lambda+6\mu)}=0\,. \tag{10.10}$$

Resolving the Eq. (10.10), de Sitter solutions are obtained. For simplicity, let us consider $n=2$, in such a case the equation (10.10) has two roots for H_0 given by,

$$H_0=\pm\frac{\sqrt{\kappa_0(3\lambda-1)}}{3\sqrt{\kappa_1(1-3\lambda+6\mu)(-1+3\lambda-2\mu)}}\,. \tag{10.11}$$

As we are interested in de Sitter points, we just consider the positive root in (10.11). Then, the stability of such de Sitter point can be analyzed by studying the derivatives of the function $F(\tilde{R})$ evaluated in H_0. The stability will depend on the value of $\frac{F_0'}{F_0''}$, which for this case yields,

$$\frac{F_0'}{F_0''} = \frac{\kappa_0}{2\kappa_1} + 12H_0^2 \,. \tag{10.12}$$

In the IR limit of the theory ($\lambda \to 1$, $\mu \to 1$), the condition for the stability of de Sitter points $\frac{F_0'}{F_0''} > 12H_0^2$ is clearly satisfied by (10.12). Even in the non IR limit, $F_0'' = 2\kappa_1$ and assuming $\kappa_1 \ll 1$, we have that,

$$\frac{F_0'}{F_0''} = 3H_0^2(1 - 3\lambda + 6\mu) + \frac{\kappa_0}{2\kappa_1} \,. \tag{10.13}$$

As this term is positive, we have that the instabilities will oscillate and be damped, such that the de Sitter point becomes stable.

Thus, we presented the example of the $F(\tilde{R})$ theory where flat space is unstable solution and de Sitter space is stable solution. The problem of scalar graviton does not appear in this theory because one has to analyze the spectrum of theory around de Sitter space which is real vacuum. Indeed, flat space is not stable and cannot be considered as the vacuum solution. Of course, deeper analysis of de Sitter spectrum structure of the theory is necessary. Nevertheless, as we see already standard $F(\tilde{R})$ gravity suggests the way to resolve the pathologies which are well-known in Hořava-Lifshitz gravity.

10.2 $U(1)$ invariant $F(\tilde{R})$ Hořava-Lifshitz gravity

Our goal is to see whether it is possible to extend the gauge symmetries for above action as in [175]. As the first step we introduce two non-dynamical fields A, B and rewrite the action for $F(\tilde{R})$ into the form

$$S_{F(\tilde{R})} = \frac{1}{\kappa^2} \int dt d^D\mathbf{x} \sqrt{g} N (B(\tilde{R} - A) + F(A)) \,. \tag{10.14}$$

It is easy to see that solving the equation of motion with respect to A, B this action reduces into the original one. On the other hand when we perform integration by parts we obtain the action in the form

$$\begin{aligned} S_{F(\tilde{R})} = \frac{1}{\kappa^2} \int dt d^D\mathbf{x} \, \big(& \sqrt{g} N B (K_{ij} \mathcal{G}^{ijkl} K_{kl} - \mathcal{V}(g) - A) \\ & + \sqrt{g} N F(A) - 2\mu \sqrt{g} N \nabla_n B K + 2\mu \partial_i B \sqrt{g} g^{ij} \partial_j N \big) \,, \end{aligned} \tag{10.15}$$

where we ignored the boundary terms and where

$$\nabla_n B = \frac{1}{N}(\partial_t B - N^i \partial_i B) \,. \tag{10.16}$$

Let us now introduce $U(1)$ symmetry where the shift function transforms as

$$\delta_\alpha N_i(\mathbf{x}, t) = N(\mathbf{x}, t) \nabla_i \alpha(\mathbf{x}, t) \,. \tag{10.17}$$

It is important to stress that as opposite to the case of pure Hořava-Lifshitz gravity the kinetic term is multiplied with B that is space-time dependent and hence it is not possible to perform similar analysis as in [175]. This procedure frequently uses the integration by parts and the fact that covariant derivative annihilates metric tensor together with the crucial assumption that N depends on time only. Now due

to the presence of B field we have to proceed step by step with the construction of the action invariant under (10.17). As the first step note that under (10.17) the kinetic term $S^{\mathrm{kin}} = \frac{1}{\kappa^2}\int dtd^D\mathbf{x}\sqrt{g}K_{ij}\mathcal{G}^{ijkl}K_{kl}$ transforms as

$$\delta_\alpha S^{\mathrm{kin}} = -\frac{2}{\kappa^2}\int dtd^D\mathbf{x}\sqrt{g}NBK_{ij}\mathcal{G}^{ijkl}\nabla_i\nabla_j\alpha\,.$$

In order to compensate this variation of the action we introduce new scalar field ν that under (10.17) transforms as

$$\delta_\alpha\nu(t,\mathbf{x}) = \alpha(t,\mathbf{x}) \tag{10.18}$$

and add to the action following term

$$S^{(1)}_\nu = \frac{2}{\kappa^2}\int dtd^D\mathbf{x}\sqrt{g}NBK_{ij}\mathcal{G}^{ijkl}\nabla_i\nabla_j\nu\,.$$

Note that under (10.18) this term transforms as

$$\delta_\alpha S^{(1)}_\nu = \frac{2}{\kappa^2}\int dtd^D\mathbf{x}\sqrt{g}NBK_{ij}\mathcal{G}^{ijkl}\nabla_k\nabla_l\alpha - \frac{2}{\kappa^2}\int dtd^D\mathbf{x}\sqrt{g}NB\nabla_i\nabla_j\alpha\mathcal{G}^{ijkl}\nabla_k\nabla_l\nu\,.$$

so that we add the second term into the action

$$S^{(2)}_\nu = \frac{1}{\kappa^2}\int dtd^D\mathbf{x}\sqrt{g}NB\nabla_i\nabla_j\nu\mathcal{G}^{ijkl}\nabla_k\nabla_j\nu\,. \tag{10.19}$$

As a result, we find that $S^{\mathrm{kin}} + S^{(1)}_\nu + S^{(2)}_\nu$ is invariant under (10.17) and (10.18).

As the next step we analyze the variation of the B-kinetic part of the action $S^{B\mathrm{kin}} = -\frac{2\mu}{\kappa^2}\int dtd^D\mathbf{x}\sqrt{g}N\nabla_nBK$ under the variation (10.17)

$$\delta_\alpha S^{B\mathrm{kin}} = \frac{2\mu}{\kappa^2}\int dtd^D\mathbf{x}\sqrt{g}\alpha\nabla^i(\nabla_iBK) + \frac{2\mu}{\kappa^2}\int dtd^D\mathbf{x}\sqrt{g}N\alpha\nabla_i\nabla_j(g^{ij}\nabla_nB)\,. \tag{10.20}$$

We see that in order to cancel this variation it is appropriate to add following expression into the action

$$\begin{aligned} S^{\nu-B} &= -\frac{2}{\kappa^2}\mu\int dtd^D\mathbf{x}\sqrt{g}N\nu\nabla^i(\nabla_iBK) - \frac{2}{\kappa^2}\mu\int dtd^D\mathbf{x}\sqrt{g}N\nu\nabla_i\nabla_j[g^{ij}\nabla_nB] \\ &\quad +\frac{2}{\kappa^2}\mu\int dtd^D\mathbf{x}\sqrt{g}N\nabla^k\nu\nabla_kB\nabla_i\nabla^i\nu\,. \end{aligned}$$

Then it is easy to see that $S^{B\mathrm{kin}} + S^{\nu-B}$ is invariant under (10.17) and (10.18). Collecting all these results we find following $F(\hat{R})$ HL action that is invariant under (10.17) and (10.18)

$$\begin{aligned} S_{F(\tilde{R})} &= \frac{1}{\kappa^2}\int dtd^D\mathbf{x}\left(\sqrt{g}NB(K_{ij}\mathcal{G}^{ijkl}K_{kl} - \mathcal{V}(g) - A)\right. \\ &\quad \left.+\sqrt{g}NF(A) - 2\mu\sqrt{g}N\nabla_nBK + 2\mu\partial_iB\sqrt{g}g^{ij}\partial_jN\right) \\ &\quad -2\mu\int d^D\mathbf{x}dt\sqrt{g}\nu\nabla^i(\nabla_iBK) - 2\mu\int d^D\mathbf{x}dt\sqrt{g}N\nu\nabla_i\nabla_j(g^{ij}\nabla_nB) \\ &\quad +2\mu\int dtd^D\mathbf{x}\sqrt{g}N\nabla^i\nu\nabla_iB\nabla^j\nabla_j\nu \\ &\quad +2\int d^D\mathbf{x}\sqrt{g}NBK_{ij}\mathcal{G}^{ijkl}\nabla_k\nabla_l\nu + \int d^D\mathbf{x}\sqrt{g}B\nabla_i\nabla_j\nu\mathcal{G}^{ijkl}\nabla_k\nabla_l\nu\,. \end{aligned} \tag{10.21}$$

Note that we can write this action in suggestive form

$$\begin{aligned}S_{F(\tilde{R})} &= \frac{1}{\kappa^2}\int dtd^D\mathbf{x}\left(\sqrt{g}NB((K_{ij}+\nabla_i\nabla_j\nu)\mathcal{G}^{ijkl}(K_{kl}+\nabla_k\nabla_l\nu)-\mathcal{V}(g)-A)\right.\\ &\left.+\sqrt{g}NF(A)-2\mu\sqrt{g}N(\nabla_nB+\nabla^i\nu\nabla_iB)g^{ij}(K_{ji}+\nabla_j\nabla_i\nu)+2\mu\partial_iB\sqrt{g}g^{ij}\partial_jN\right)\end{aligned} \tag{10.22}$$

or in even more suggestive form by introducing

$$\bar{N}_i = N_i - N\nabla_i\nu\,, \quad \bar{K}_{ij} = \frac{1}{2N}(\partial_t g_{ij} - \nabla_i\bar{N}_j - \nabla_j\bar{N}_i) \tag{10.23}$$

so that

$$\begin{aligned}S_{F(\tilde{R})} &= \frac{1}{\kappa^2}\int dtd^D\mathbf{x}\left(\sqrt{g}NB(\bar{K}_{ij}\mathcal{G}^{ijkl}\bar{K}_{kl}-\mathcal{V}(g)-A)\right.\\ &\left.+\sqrt{g}NF(A)-2\mu\sqrt{g}N\hat{\nabla}_nBg^{ij}\bar{K}_{ji}+2\mu\partial_iB\sqrt{g}g^{ij}\partial_jN\right)\end{aligned} \tag{10.24}$$

is formally the same as the $F(\tilde{R})$ Hořava-Lifshitz gravity action. Clearly this action is invariant under arbitrary $\alpha=\alpha(t,\mathbf{x})$. Moreover, such an introduction of $U(1)$ symmetry is trivial and does not modify the physical properties of the theory. This is nicely seen from the fact that ν appears in the action in the combination with N_i through $\bar{N}_i$ where ν plays the role of the Stückelberg field. In order to get physical content of given symmetry we follow [175] and [113] and introduce following term into action

$$S^{\nu,k} = \frac{1}{\kappa^2}\int dtd^D\mathbf{x}\sqrt{g}B\mathcal{G}(g_{ij})(\mathcal{A}-a)\,, \tag{10.25}$$

where

$$a = \dot{\nu} - N^i\nabla_i\nu + \frac{N}{2}\nabla^i\nabla_i\nu\,, \tag{10.26}$$

where $\dot{X}\equiv\frac{dX}{dt}$. Note a transforms under α variation as

$$a'(t,\mathbf{x}) = a(t,\mathbf{x}) + \dot{\alpha}(t,\mathbf{x}) - N^i(t,\mathbf{x})\nabla_i\alpha(t,\mathbf{x})\,.$$

Then it is natural to suppose that $\mathcal{A}$ transforms under α-variation as

$$\mathcal{A}'(t,\mathbf{x}) = \mathcal{A}(t,\mathbf{x}) + \dot{\alpha}(t,\mathbf{x}) - N^i(t,\mathbf{x})\nabla_i\alpha(t,\mathbf{x}) \tag{10.27}$$

so that we immediately see that $\mathcal{A}-a$ is invariant under α-variation. The function $\mathcal{G}$ can generally depend on arbitrary combinations of metric g and matter field and we only demand that it should be invariant under foliation preserving diffeomorphism and under (10.17) and (10.18). For our purposes it is, however, sufficient to restrict ourselves to the models where $\mathcal{G}$ depends on the spatial curvature R only. Now one observes that the equation of motion for $\mathcal{A}$ implies the constraint

$$B\mathcal{G}(R) = 0 \tag{10.28}$$

that for non-zero B implies the condition $\mathcal{G}(R)=0$. Note that this condition is crucial for elimination of the scalar graviton when we study fluctuations around flat background. We demonstrate this important result in the next section.

Finally note that it is possible to integrate out B and A fields from the actions (10.24) and (10.25) that leads to

$$\begin{aligned}S_{F(\bar{R})} &= \frac{1}{\kappa^2}\int dtd^D\mathbf{x}\sqrt{g}NF(\bar{R})\,,\\ \bar{R} &= \bar{K}_{ij}\mathcal{G}^{ijkl}\bar{K}_{kl}-\mathcal{V}(g)+\frac{2\mu}{\sqrt{g}N}\left\{\partial_t\left(\sqrt{g}\bar{K}\right)-\partial_i\left(\sqrt{g}N^i\bar{K}\right)\right\}+\frac{1}{N}\mathcal{G}(\mathcal{A}-a)\,.\end{aligned} \tag{10.29}$$

This finishes the construction of $U(1)$ invariant $F(\bar{R})$ HL theory action.

10.2.1 Lagrangian for the scalar field

Now we extend above analysis to the action for the matter field with the following general form of the scalar field action

$$S_{\mathrm{matt}} = -\int dt d^D\mathbf{x}\sqrt{g} N X\,, \tag{10.30}$$

where

$$X = -(\nabla_n\phi)^2 + F(g^{ij}\partial_i\phi\partial_j\phi)\,. \tag{10.31}$$

where $F(x) = X + \sum_{n=2}^{z} X^n$ and where we defined

$$\nabla_n\phi = \frac{1}{N}(\partial_t\phi - N^i\partial_i\phi)\,. \tag{10.32}$$

Note that this general form of the scalar field action is consistent with the anisotropy of target space-time as was shown in [73, 184, 185, 208, 182, 261].

Now we try to extend above action in order to make it invariant under (10.17). Note that under this variation the scalar field action (10.30) transforms as

$$\delta_\alpha S_{\mathrm{matt}} = -2\int dt d^D\mathbf{x}\sqrt{g} N\nabla^i\alpha\nabla_i\phi\nabla_n\phi \tag{10.33}$$

using

$$\delta_\alpha X = 2\nabla^i\alpha\nabla_i\phi\nabla_n\phi\,. \tag{10.34}$$

We compensate the variation (10.33) by introducing additional term into action

$$S_{\mathrm{matt}-\nu} = -2\int dt d^D\mathbf{x}\sqrt{g}\nu N\nabla^i(\nabla_i\phi\nabla_n\phi) + \int dt d^D\mathbf{x}\sqrt{g}\nabla^i\nu\nabla^j\nu\nabla_i\phi\nabla_j\phi \tag{10.35}$$

Then the action (10.35) transforms under (10.17) and (10.18) as

$$\delta_\alpha S_{\mathrm{matt}-\nu} = 2\int dt d^D\mathbf{x}\sqrt{g} N\nabla^i\alpha\nabla_i\phi\nabla_n\phi \tag{10.36}$$

that compensates the variation (10.33).

In the same way one can analyze more general form of the scalar action

$$S_{\mathrm{matt}} = -\int dt d^D\mathbf{x}\sqrt{g} N K(X)\,. \tag{10.37}$$

In order to find the generalization of the action (10.37) which is invariant under (10.17) we introduce two auxiliary fields C, D and write the action (10.37) as

$$S_{\mathrm{matt}} = -\int dt d^D\mathbf{x}\sqrt{g} N[K(C) + D(X - C)]\,. \tag{10.38}$$

Clearly this action transforms under (10.17) as

$$\delta_\alpha S_{\mathrm{matt}} = -\int dt d^D\mathbf{x}\sqrt{g} N D\delta_\alpha X = -2\int dt d^D\mathbf{x}\sqrt{g} N D\nabla^i\alpha\nabla_i\phi\nabla_n\phi\,, \tag{10.39}$$

where relation (10.34) is used. It is easy to see that the variation of the following term

$$S_{\mathrm{matt}-\nu} = 2\int dtd^D\mathbf{x}\sqrt{g}ND\nabla^i\nu\nabla_i\phi\nabla_n\phi + \int dtd^D\mathbf{x}\sqrt{g}D\nabla^i\nu\nabla^j\nu\nabla_i\phi\nabla_j\phi \tag{10.40}$$

compensates the variation (10.39). Finally note that (10.38) together with (10.40) can be written in more elegant form

$$S_{\mathrm{matt}} = -\int dtd^D\mathbf{x}\sqrt{g}N[K(C) + D(\bar{X} - C)] = -\int dtd^D\mathbf{x}\sqrt{g}NK(\bar{X}) , \tag{10.41}$$

where

$$\begin{aligned} \bar{X} &= -(\bar{\nabla}_n\phi)^2 + F(g^{ij}\partial_i\phi\partial_j\phi) , \\ \bar{\nabla}_n\phi &= \frac{1}{N}(\partial_t\phi - \bar{N}^i\nabla_i\phi) = \frac{1}{N}(\partial_t\phi - N^i\nabla_i + N\nabla^i\nu\nabla_i\phi) . \end{aligned}$$

In this section we constructed $U(1)$-invariant scalar field action in the form which closely follows the original construction presented in [175]. In the next section more elegant approach to the construction of $U(1)$ invariant $F(\bar{R})$ HL gravity and the scalar field action is given.

10.3 Study of fluctuations around flat background in $U(1)$ invariant $F(\tilde{R})$ HL Gravity

Let us analyze the spectrum of fluctuations in case of $U(1)$ invariant $F(\tilde{R})$ HL gravity for the special case $\mu = 0$. For simplicity we assume that $F(\bar{R})$ Hořava-Lifshitz gravity has flat space-time as its solution with the background values of the fields

$$g_{ij}^{(0)} = \delta_{ij} , \quad N^{(0)} = 1 , \quad N_i^{(0)} = 0 , \quad \mathcal{A}^{(0)} = 0 , \quad \nu^{(0)} = 0 . \tag{10.42}$$

Note that for the flat background the equation of motion for B and A takes the form

$$\mathcal{V}(g^{(0)}) - A^{(0)} = 0 , \quad B^{(0)} - F'(A^{(0)}) = 0 \tag{10.43}$$

that implies that $A^{(0)}$, $B^{(0)}$ are constants. In order to find the spectrum of fluctuations we expand all fields up to linear order around this background

$$\begin{aligned} &g_{ij} = \delta_{ij} + \kappa h_{ij} , \quad N_i = \kappa n_i , \quad N = 1 + \kappa n , \\ &A = A^{(0)} + \kappa a , \quad B = B^{(0)} + \kappa b , \quad \mathcal{A} = \mathcal{A}^{(0)} + \kappa\tilde{\mathcal{A}} , \quad \nu = \nu^{(0)} + \kappa\tilde{\nu} . \end{aligned}$$

Since n depends on t only, its equation of motion gives one integral constraint. This constraint does not affect the number of local degrees of freedom. For that reason it is natural to consider the equation of motion for h_{ij}, n_i and ν only. We further decompose the field h_{ij} and n_i into their irreducible components

$$h_{ij} = s_{ij} + \partial_i w_j + \partial_j w_i + \left(\partial_i\partial_j - \frac{1}{D}\delta_{ij}\partial^2\right) M + \frac{1}{D}\delta_{ij}h , \tag{10.44}$$

where the scalar $h = h_{ii}$ is the trace part of h_{ij} while s_{ij} is symmetric, traceless and transverse

$$\partial^i s_{ij} = 0 , \partial^i = \delta^{ij}\partial_j \tag{10.45}$$

and w_i is transverse

$$\partial^i w_i = 0 . \tag{10.46}$$

In the same way we decompose n_i

$$n_i = u_i + \partial_i C \tag{10.47}$$

with u_i transverse $\partial^i u_i = 0$. In what follows we fix the spatial diffeomorphism symmetry by fixing the gauge

$$w_i = 0\,, \quad M = 0\,. \tag{10.48}$$

We begin with the equation of motion for N_i

$$2\nabla_j(B\mathcal{G}^{ijkl}\bar{K}_{kl}) + B\mathcal{G}(R)\nabla_i\nu = 0\,. \tag{10.49}$$

and for ν

$$2\nabla_i\nabla_j(BN\mathcal{G}^{ijkl}\bar{K}_{kl}) + \frac{1}{\sqrt{g}}\left(\frac{d}{dt}(\sqrt{g}B\mathcal{G}) - \nabla_i(\sqrt{g}BN^i\mathcal{G}) - \frac{1}{2}\nabla_i\nabla^i(\sqrt{g}BN\mathcal{G})\right) = 0\,.$$

Combining these equations and using the equation of motion for $\mathcal{A}$ one obtains

$$\mathcal{G}(R) = 0 \tag{10.50}$$

Then

$$B\frac{d}{dt}\mathcal{G} - BN^i\nabla_i\mathcal{G} - \frac{N}{2}(B\nabla_i\nabla^i\mathcal{G} + 2\nabla_i B\nabla^i\mathcal{G}) = 0\;. \tag{10.51}$$

Further, in the linearized approximation the equation of motion (10.49) takes the form

$$-2B_0(1-\lambda)\partial^i(\partial_k\partial^k C) + B_0\partial_k\partial^k u^i + \frac{1}{D}(1-D\lambda)\partial^i\dot{h} + 2(1-\lambda)B_0\partial^i(\partial_j\partial^j\tilde{\nu}) = 0 \tag{10.52}$$

using the fact that $\mathcal{G}(R_0) = 0$ and also $\partial^i s_{ij} = \partial^j s_{ij} = 0$ and $\delta^{ij}s_{ji} = 0$. Let us now focus on the solution of the constraint $\mathcal{G}(R) = 0$ in the linearized approximation. Let $R_0^{(D)}$ is the solution of the equation of motion and let us consider the perturbation around this equation. These perturbations have to obey the equation

$$\mathcal{G}(R_0 + \delta R) = \mathcal{G}(R_0) + \frac{d\mathcal{G}}{dR}\delta R = \frac{d\mathcal{G}}{dR}(R_0)\delta R = 0\,.$$

We see that in order to eliminate the scalar graviton we have to demand that $\frac{d\mathcal{G}}{dR}(R_0) \neq 0$. To proceed further note that

$$\delta R_{ij} = \frac{1}{2}[\nabla_i^{(0)}\nabla^{(0)k}h_{jk} + \nabla_j^{(0)}\nabla^{k(0)}h_{ik} - \nabla_k^{(0)}\nabla^{k(0)}h_{ij} - \nabla_i^{(0)}\nabla_j^{(0)}h] \tag{10.53}$$

where $\nabla^{(0)}$ is the covariant derivative calculated using the background metric $g_{ij}^{(0)}$. Then in the flat background it follows

$$\delta R = \frac{\kappa}{D}(1-D)\partial^k\partial_k h$$

so that the condition $\delta R = 0$ implies $\partial_k\partial^k h = 0$. Then $h = h(t)$. However, in this case the fluctuation mode does not obey the boundary conditions that we implicitly assumed. Explicitly we demand that all fluctuations vanish at spatial infinity. For that reason one should demand that $h = 0$. Then it is easy to see that the equation (10.51) is trivially solved. In the linearized approximation the equation of motion for g_{ij} takes the form

$$\begin{aligned}&\frac{1}{2}(\ddot{s}_{ij} + \frac{1}{D}\delta_{ij}(1-\lambda D)\ddot{h} - \partial_i\dot{u}_j - 2\partial_i\partial_j\dot{C} + 2\partial^i\partial^j\tilde{\nu} - 2\lambda\delta^{ij}\partial_k\partial^k\tilde{\nu} + 2\lambda\delta_{ij}\partial_k\partial^k\dot{C})\\ &-\frac{\delta\mathcal{V}_2}{g_{ij}} + \frac{d\mathcal{G}}{dR}[\partial^i\partial^j(\tilde{\mathcal{A}} - \tilde{a}) + \delta^{ij}\partial_k\partial^k(\tilde{\mathcal{A}} - \tilde{a})] = 0\end{aligned} \tag{10.54}$$

where

$$\tilde{a} = \frac{d\tilde{\nu}}{dt} + \frac{1}{2}\partial_k\partial^k\tilde{\nu}\,. \tag{10.55}$$

Note that we have not fixed the $U(1)$ gauge symmetry yet. It turns out that it is natural to fix it as

$$\nu = 0\,. \tag{10.56}$$

Then the trace of the equation (10.54) is equal to

$$(1-\lambda D)\partial_k\partial^k\dot{C} + \delta^{ij}\frac{\delta\mathcal{V}_2}{\delta g_{ij}} - \frac{d\mathcal{G}}{dR}(1-D)\partial_i\partial^i\tilde{\mathcal{A}} = 0\,. \tag{10.57}$$

Let us again consider the equation of motion (10.52) and take its ∂^i. Then using the fact that $\partial^i u_i = 0$ one gets the condition $C = f(t)$ that again with suitable boundary conditions implies $C = 0$. However then inserting this result in (10.52) we find

$$\partial_k\partial^k u_i = 0 \tag{10.58}$$

that also implies $u_i = 0$.

To proceed further we now assume that $\mathcal{V}_2 = -R$ so that $\frac{\delta\mathcal{V}_2}{\delta g_{ij}} = -\frac{1}{2}\partial_k\partial^k s_{ij} - \frac{D-2}{2D}(\partial_i\partial_j - \delta_{ij}\partial^k)h$. Clearly the trace of this equation is proportional to h and hence it implies following equation for $\tilde{a}$

$$\partial_k\partial^k\tilde{\mathcal{A}} = 0\,. \tag{10.59}$$

Imposing again the requirement that $\tilde{\mathcal{A}}$ vanishes at spatial infinity we find that the only solution of given equation is $\tilde{\mathcal{A}} = 0$. Finally the equation of motion for g_{ij} gives following result

$$\ddot{s}_{ij} + \partial_k\partial^k s_{ij} = 0\,. \tag{10.60}$$

In other words, it is demonstrated that under assumption that $U(1)$ invariant $F(\bar{R})$ HL gravity has flat space-time as its solution it follows that the perturbative spectrum contains the transverse polarization of the graviton only. Clearly, this result may be generalized for general version of theory with arbitrary parameter μ.

Finally we consider the linearized equations of motion for A and B. In case of A one gets

$$-b + F''(A_0)a = 0 \tag{10.61}$$

while in case of B we obtain

$$-\frac{d\mathcal{V}}{dR}(R_0)\delta R - a = 0 \tag{10.62}$$

Using the fact that $\delta R \sim h = 0$ we get from (10.62) and from (10.61)

$$a = b = 0\,. \tag{10.63}$$

In other words there are no fluctuations corresponding to the scalar fields A and B. One can compare this situation with the conventional $F(R)$ gravity where the mathematical equivalence of the theory with the Brans-Dicke theory implies the existence of propagating scalar degrees of freedom. In our case, however, the fact that $U(1)$ invariant $F(\bar{R})$ HL gravity is invariant under the foliation preserving diffeomorphism allows us to consider theory without kinetic term for B ($\mu = 0$).

Thus, it seems U(1) extension of $F(R)$ HL gravity may lead to solution of the problem of scalar graviton.

10.4 Cosmological Solutions of $U(1)$ Invariant $F(\tilde{R})$ HL gravity

Let us investigate the flat FLRW cosmological solutions for the theory described by action for $F(\tilde{R})$. Spatially-flat FLRW metric is now assumed, $ds^2 = -N^2 dt^2 + a^2(t) \sum_{i=1}^{3} \left(dx^i\right)^2$, and we choose the gauge fixing condition for the local $U_\Sigma(1)$ symmetry as follows,

$$\nu = 0 \,. \tag{10.64}$$

In the flat FLRW metric, one gets $N_i = 0$ and $\bar{R}$ and $\bar{K}_{ij}$ only depend on the cosmological time t. Therefore, the constraint equation can be satisfied trivially. The FLRW equations are identical with the corresponding equations in the $F(\tilde{R})$-gravity in absence of $U(1)$ symmetry given by (8.13) and (8.15) . Hence, $U(1)$ extension does not influence the FLRW cosmological dynamics. Let us consider the theory which admits a de Sitter universe solution. We now neglect the matter contribution by putting $p_m = \rho_m = 0$. Then by assuming $H = H_0$, first FLRW equation (8.15) gives

$$0 = F\left(3\left(1-3\lambda+6\mu\right)H_0^2\right) - 6\left(1-3\lambda+3\mu\right)H_0^2 F'\left(3\left(1-3\lambda+6\mu\right)H_0^2\right) \,, \tag{10.65}$$

as long as the integration constant vanishes ($C = 0$) in (8.15). We now consider the following model:

$$F\left(\bar{R}\right) \propto \bar{R} + \beta \bar{R}^2 + \gamma \bar{R}^3 \,. \tag{10.66}$$

Then Eq. (10.65) becomes

$$\begin{aligned} 0 &= H_0^2 \left\{1 - 3\lambda + 9\beta\left(1-3\lambda+6\mu\right)\left(1-3\lambda+2\mu\right)H_0^2 \right. \\ &\quad \left. +9\gamma\left(1-3\lambda+6\mu\right)^2\left(5-15\lambda+12\mu\right)H_0^4\right\} \,, \end{aligned} \tag{10.67}$$

which has the following two non-trivial solutions,

$$H_0^2 = -\frac{\left(1-3\lambda+2\mu\right)\beta}{2\left(1-3\lambda+6\mu\right)\left(5-15\lambda+12\mu\right)\gamma}\left(1 \pm \sqrt{1 - \frac{4\left(1-3\lambda\right)\left(5-15\lambda+12\mu\right)\gamma}{9\left(1-3\lambda+2\mu\right)^2\beta^2}}\right) \,, \tag{10.68}$$

as long as the r.h.s. is real and positive. If

$$\left|\frac{4\left(1-3\lambda\right)\left(5-15\lambda+12\mu\right)\gamma}{9\left(1-3\lambda+2\mu\right)2\beta^2}\right| \ll 1 \,, \tag{10.69}$$

one of the two solutions is much smaller than the other solution. Then one may regard that the larger solution corresponds to the inflation in the early universe and the smaller one to the late-time acceleration.

More examples of $F\left(\bar{R}\right)$ theory which can contain more than one dS solution, such that inflation and dark energy epochs can be explained under the same mechanism, as it was pointed in the above chapters, may be considered. First of all, as generalization of the model (10.66), a general polynomial function may be discussed

$$F\left(\bar{R}\right) = \sum_{n=1}^{m} \alpha_n \bar{R}^n \,, \tag{10.70}$$

Here α_n are coupling constants. Using the equation (10.65), it yields the algebraic equation,

$$0 = \sum_{n=1}^{m} \alpha_n \bar{R}_0^n - 2\frac{1-3\lambda+3\mu}{1-3\lambda+6\mu}\bar{R}_0 \sum_{n=1}^{m} n\alpha_n \bar{R}_0^{n-1} \,. \tag{10.71}$$

By a qualitative analysis, one can see that the number of positive real roots, i.e., of the de Sitter points, depends completely on the sign of the coupling constants α_n . Then, by a proper choice, $F\left(\bar{R}\right)$ gravity can well explain dark energy and inflationary epochs in a unified natural way. Even it could predict the existence of more than two accelerated epochs, which could resolve the coincidence problem.

Let us now consider an explicit example

$$F\left(\bar{R}\right) = \frac{\bar{R}}{\bar{R}(\alpha\bar{R}^{n-1} + \beta) + \gamma}\,, \tag{10.72}$$

where α, β, γ, n are constants. By introducing this function in (10.65), it is straightforward to show that for the function (10.72), there are several de Sitter solutions. In order to simplify this example, let us consider the case $n = 2$, where the equation (10.65) yields,

$$\gamma - 3\gamma\lambda - 3\beta H_0^2(1 - 3\lambda + 6\mu)^2 + 27\alpha H_0^4(-1 + 3\lambda - 4\mu)(1 - 3\lambda + 6\mu)^2 = 0\,. \tag{10.73}$$

The solutions are given by

$$\begin{aligned} H_0^2 = &\frac{1}{18\alpha(1 - 3\lambda + 6\mu)^2(-1 + 3\lambda - 4\mu)}\left\{\beta(1 - 3\lambda + 6\mu)^2\right. \\ &\left.\pm\sqrt{(1 - 3\lambda + 6\mu)^2\left[12\alpha\gamma(-1 + 3\lambda)(-1 + 3\lambda - 4\mu) + \beta^2(1 - 3\lambda + 6\mu)^2\right]}\right\}\,. \end{aligned} \tag{10.74}$$

Then, by a proper choice of the free parameters of the model, two positive roots of the equation (10.73) are solutions. Hence, such a model can explain inflationary and dark energy epochs in unified manner.

10.5 Discussion

In summary, in this chapter we aimed to resolve (at least, partially) the inconsistency problems of the projectable HL gravity. First of all, it is demonstrated that some versions of $F(R)$ HL gravity may have stable de Sitter solution and unstable flat space solution. As a result, the spectrum analysis showing the presence of scalar graviton is not applied. The whole spectrum analysis should be redone for de Sitter background.

Second, $U(1)$ extension of $F(R)$ HL gravity is formulated. The analysis of fluctuations of $U(1)$ invariant $F(\bar{R})$ HL gravity is performed. It is shown that like in case of $U(1)$ HL gravity the scalar graviton ghost does not emerge. This opens good perspectives for consistency of such class of models. It is also interesting that spatially-flat FLRW equations for $U(1)$ invariant $F(\bar{R})$ gravity turn out to be just the same as for the one without $U(1)$ symmetry. This indicates that all (spatially-flat FLRW) cosmological predictions of viable conventional $F(R)$ gravity are just the same as for its HL counterpart (with special parameters choice).

Part IV

Other aspects of FLRW Universes

Chapter 11

Generalizing Cardy-Verlinde formula and entropy bounds near future singularities

[1]In the above chapters, several alternatives to explain the dark energy epoch have been analised, and even the inflationary phase by means mainly of modifications of gravity, which can be modelled as effective fluids. The study of phantom-like or quintessence-like effective dark fluids opens for a number of new phenomena which are typical for such a DE universe. For instance, it is known that phantom DE may drive the future universe to a so-called finite-time Big Rip singularity (for earlier works on this, see Refs.[12, 13, 15, 22, 67, 70, 72, 103, 104, 114, 116, 133, 137, 145, 155, 159, 160, 164, 170, 202, 203, 220, 218, 274, 279, 294, 311]). From another side, quintessence-like DE may bring the future universe to a milder future singularity (like the sudden singularity [17, 20, 21, 23, 25, 39, 38, 40, 77, 98, 112, 151, 152, 194, 224, 222, 236, 273, 295, 290] where the effective energy-density is finite). Actually, the study of Ref.[246] shows that there are four different types of future finite-time singularities where the Type I singularity corresponds to the Big Rip, the sudden singularity is of Type II, etc. The universe looks quite strange near to the singularity where curvature may grow up so that quantum gravity effects may be dominant [133]. In any case, the study of the universe under critical conditions (for instance, near a future singularity) may clarify the number of fundamental issues relating seemingly different physical theories.

Some time ago [297] it was shown that the first FLRW equation for a closed FLRW universe may have a more fundamental origin than what is expected from standard General Relativity. It was demonstrated that this equation may be rewritten so as to describe the universe entropy in terms of total energy and Casimir energy (the so-called Cardy-Verlinde (CV) formula). Moreover, it turns out that the corresponding formula has a striking correspondence with the Cardy formula for the entropy of a two-dimensional conformal field theory (2d CFT). Finally, the formula may be rewritten as a dynamical entropy bound from which a number of entropy bounds, proposed earlier, follow. The connection between the standard gravitational equation and the 2d CFT dynamical entropy bound indicates a very deep relation between gravity and thermodynamics. It raises the question about to which extent the CV formula is universal. Problems of this sort are natural to study when the universe is under critical conditions, such as near a singularity.

The present chapter is devoted to a study of the universality of the CV formula and the corresponding

[1]This Chapter is based on: [58]

dynamical entropy bound in a DE universe filled with a generalized fluid, especially near the singularity regime. Generalization of the CV formula for a multicomponent fluid with interactions, assuming the EoS to be inhomogeneous, is presented. It is shown that the standard CV formula with correct power (square root) is restored only for some very special cases. The dynamical entropy bound for such fluids near all the four types of the future singularity is considered. It is demonstrated that this dynamical entropy bound is most likely violated near the singularity (except from some cases of Type II and Type IV singularity). This situation is not qualitatively changed even if account is taken of quantum effects in conformally invariant theory. Using the formalism of modified gravity to describe an effective dark fluid, the corresponding CV formula is constructed also for $F(R)$-gravity. The corresponding dynamical entropy bound is derived. It is shown that the bound is satisfied for a de Sitter universe solution. Further discussion and outlook is given in the discussion section.

11.1 Generalization of Cardy-Verlinde formula in FLRW Universe for various types of fluids

This section is devoted to consideration of the Cardy-Verlinde formula for more general scenarios than those considered in previous works (see [297, 308]). We consider here a $(n+1)$-dimensional spacetime described by the FLRW metric, written in comoving coordinates as

$$ds^2 = -dt^2 + \frac{a(t)^2 dr^2}{1-kr^2} + r^2 d\Omega_{n-1}^2 \ , \tag{11.1}$$

where $k=-1,0,+1$ for an open, flat, or closed spatial Universe, $a(t)$ is taken to have units of lenght, and $d\Omega_{n-1}^2$ is the metric of an $n-1$ sphere. By inserting the metric (11.1) in the Einstein field equations the FLRW equations are derived,

$$H^2 = \frac{16\pi G}{n(n-1)} \sum_{i=1}^{m} \rho_i - \frac{k}{a^2} \ , \quad \dot{H} = -\frac{8\pi G}{n-1} \sum_{i=1}^{m} (\rho_i + p_i) + \frac{k}{a^2} \ . \tag{11.2}$$

Here $\rho_i = E_i/V$ and p_i are the energy-density and pressure of the matter component i that fills the Universe. In this chapter we consider only the $k=1$ closed Universe. Moreover, we assume an equation of state (EoS) of the form $p_i = w_i \rho_i$ with w_i constant for each fluid, and assume at first no interaction between the different components. Then, the conservation law for energy has the form

$$\dot{\rho}_i + nH\left(\rho_i + p_i\right) = 0 \ , \tag{11.3}$$

and by solving (11.3), we find that the i fluid depends on the scale factor as

$$\rho_i \propto a^{-n(1+w_i)} \ . \tag{11.4}$$

Let us now review the case of Ref.[308], where just one fluid with EoS $p = w\rho$ and $w = \text{constant}$ is considered. The total energy inside the comoving volume V, $E = \rho V$, can be written as the sum of an extensive part E_{E} and a subextensive part E_{C}, called the Casimir energy, and takes the form:

$$E(S,V) = E_{\text{E}}(S,V) + \frac{1}{2} E_{\text{C}}(S,V) \ . \tag{11.5}$$

Under a rescaling of the entropy ($S \to \lambda S$) and the volume ($V \to \lambda V$), the extensive and subextensive parts of the total energy transform as

$$E_{\text{E}}(\lambda S, \lambda V) = \lambda E_{\text{E}}(S,V) \ , \quad E_{\text{C}}(\lambda S, \lambda V) = \lambda^{1-2/n} E_{\text{C}}(S,V) \ . \tag{11.6}$$

Hence, by assuming that the Universe satisfies the first law of thermodynamics, the term corresponding to the Casimir energy $E_{\rm C}$ can be seen as a violation of the Euler identity according to the definition in Ref.[297]:

$$E_{\rm C} = n(E + pV - TS) \ . \tag{11.7}$$

Since the total energy behaves as $E \sim a^{-nw}$ and by the definition (11.5), the Casimir energy also goes as $E_{\rm C} \sim a^{-nw}$. The FLRW Universe expands adiabatically, $dS = 0$, so the products $E_{\rm C}a^{nw}$ and $E_{\rm E}a^{nw}$ should be independent of the volume V, and be just a function of the entropy. Then, by the rescaling properties (11.6), the extensive and subextensive part of the total energy can be written as functions of the entropy only [308],

$$E_{\rm E} = \frac{\alpha}{4\pi a^{nw}} S^{w+1} \ , \quad E_{\rm C} = \frac{\beta}{2\pi a^{nw}} S^{w+1-2/n} \ . \tag{11.8}$$

Here α and β are undetermined constants. By combining these expressions with (11.5), the entropy of the Universe is written as a function of the total energy E and the Casimir energy $E_{\rm C}$, [308],

$$S = \left(\frac{2\pi a^{nw}}{\sqrt{\alpha\beta}} \sqrt{E_{\rm C}(2E - E_{\rm C})} \right)^{\frac{n}{n(w+1)-1}} \ , \tag{11.9}$$

which for $w = 1/n$ (radiation-like fluid) reduces to [297],

$$S = \frac{2\pi a}{\sqrt{\alpha\beta}} \sqrt{E_{\rm C}(2E - E_{\rm C})} \ , \tag{11.10}$$

which has the same form as the Cardy formula given in Ref.[89]. The first FLRW equation (11.2) can be rewritten as a relation between thermodynamics variables, and yields

$$S_{\rm H} = \frac{2\pi}{n} a \sqrt{E_{\rm BH}(2E - E_{\rm BH})} \ , \quad \text{where} \quad S_{\rm H} = (n-1)\frac{HV}{4G} \ , \quad E_{\rm BH} = n(n-1)\frac{V}{8\pi G a^2} \ . \tag{11.11}$$

It is easy to check that for the bound proposed in Ref.[297], $E_{\rm C} \leq E_{\rm BH}$, the equation for the entropy (11.10) coincides with the first FLRW equation (11.11) when the bound is reached. We will see below that when there are several fluid components, the same kind of expression as in Ref.[297] cannot be found. Nor is there the same correspondence with the FLRW equation when the bound is saturated.

Multicomponent Universe

If m fluids are considered with arbitrary EoS, $p_i = w_i \rho_i$, the expression for the total entropy is simple to derive just by following the same method as above. The total entropy is given by the sum of the entropies for each fluid,

$$S = \sum_{i=1}^{m} S_i = \sum_{i=1}^{m} \left(\frac{2\pi a^{nw_i}}{\sqrt{\alpha\beta}} \sqrt{E_{iC}(2E_i - E_{iC})} \right)^{\frac{n}{n(w_i+1)-1}} \ . \tag{11.12}$$

This expression cannot be reduced to one depending only on the total energy unless very special conditions on the nature of the fluids are assumed. Let us for simplicity assume that there are only two fluids with EoS given by $p_1 = w_1\rho_1$ and $p_2 = w_2\rho_2$, w_1 and w_2 being constants. We can substitute the fluids by an effective fluid described by the EoS

$$p_{\rm eff} = w_{\rm eff}\rho_{\rm eff} \ , \quad \text{where} \quad w_{\rm eff} = \frac{p_1 + p_2}{\rho_1 + \rho_2} = w_1 + \frac{w_2 - w_1}{1 + \rho_1/\rho_2} \ , \tag{11.13}$$

and $p_{\rm eff} = \frac{1}{2}(p_1 + p_2)$, $\rho_{\rm eff} = \frac{1}{2}(\rho_1 + \rho_2)$. Then, by using the energy conservation equation (11.3), we find $\rho_1 \sim (a/a_0)^{-n(1+w_1)}$ and $\rho_2 \sim (a/a_0)^{-n(1+w_2)}$, where a_0 is assumed to be the value of the scale factor at the time t_0 . The effective EoS parameter $w_{\rm eff}$ can be expressed as a function of the scale factor $a(t)$

$$w_{\rm eff} = w_1 + \frac{w_2 - w_1}{1 + (a/a_0)^{n(w_2 - w_1)}} \ . \tag{11.14}$$

The total energy inside a volume V becomes

$$E_T = E_1 + E_2 \propto (a/a_0)^{-nw_1} + (a/a_0)^{-nw_2} \ . \tag{11.15}$$

As the energy is proportional to two different powers of the scale factor a , it is not possible to write it as a function of the total entropy only. As a special case, if the EoS parameters are $w_1 = w_2 = w_{\rm eff}$, the formula for the entropy reduces to (11.9), and coincides with the CV formula when $w_{\rm eff} = 1/n$.

As another case, one might consider that for some epoch of the cosmic history, $w_1 \gg w_2$. Taking also $a >> a_0$, we could then approximate the total energy by the function $E_T \propto a^{-nw_2}$. From (11.5) the Casimir energy would also depend on the same power of a , $E_{\rm C} \propto a^{-nw_2}$. The expression (11.9) is again recovered with $w = w_2$.

Thus in general, when a multicomponent FLRW Universe is assumed, the formula for the total entropy does not resemble the Cardy formula, nor does it correspond to the FLRW equation when the Casimir bound is reached. It becomes possible to reconstruct the formula (11.10), and establish the correspondence with the Cardy formula, only if we make specific choices for the EoS of the fluids.

Interacting fluids

As a second case we now consider a Universe, described by the metric (11.1), filled with two interacting fluids. One can write the energy conservation equation for each fluid as

$$\dot{\rho_1} + nH(\rho_1 + p_1) = Q \ , \quad \dot{\rho_2} + nH(\rho_2 + p_2) = -Q \ , \tag{11.16}$$

where Q is a function that accounts for the energy exchange between the fluids. This kind of interaction has been discussed previously in studies of dark energy and dark matter. The effective EoS parameter is given by the same expression (11.13) as before. With a specific choice for the coupling function Q , the equations (11.16) may be solved. One can in principle find the dependence of the energy densities $\rho_{1,2}$ on the scale factor a ,

$$\rho_1 = a(t)^{-n(1+w_1)} \left(C_1 + \int a^{n(1+w_1)} Q(t) dt \right) \ , \quad \rho_2 = a(t)^{-n(1+w_2)} \left(C_2 - \int a^{n(1+w_2)} Q(t) dt \right) \ , \tag{11.17}$$

where C_1 and C_2 are integration constants. In general it is not possible to reproduce the CV formula, and the result will be a sum of different contributions, similar to the entropy expression given in (11.12). However, for the case where the effective EoS parameter (11.14) is a constant, the expression for the entropy will be given by equation (11.9) as before. This condition only holds when $w_{\rm eff} = w_1 = w_2$, where the situation is thus equivalent to the one-fluid case, and the entropy reduces to the CV formula when $w_{\rm eff} = 1/n$.

Let us consider a simple choice for the function Q that leads to the CV formula for a certain limit. Let $Q = Q_0 a^m H$, where m is a positive number, Q_0 is a constant, and $H(t)$ the Hubble parameter. Then

the integral in (11.17) is easily calculated, and the energy densities depend on the scale factor according to

$$\rho_1 = C_1 a^{-n(1+w_1)} + k_1 a^m \ , \quad \rho_2 = C_2 a^{-n(1+w_2)} + k_2 a^m \ , \tag{11.18}$$

where $k_{1,2} = Q_0/(n(1+w_{1,2})+m)$. If we restrict ourselves to the regime where $a \gg C_{1,2}$ such that the first terms in the expressions for $\rho_{1,2}$ are negligible, the effective EoS parameter becomes

$$w_{\text{eff}} = w_1 + \frac{w_2 - w_1}{1 + \frac{k_1}{k_2}} \ . \tag{11.19}$$

Then, the entropy of the universe is given by (11.9) with $w = w_{\text{eff}}$. The CV formula can be reproduced only with very specific choice of the free parameters, just as above.

We have thus shown that in general a formula for the entropy of the type (11.9) cannot be reconstructed for interacting fluids. Coincidence with the Cardy formula is obtained if the effective EoS parameter is radiation-like, $w_{\text{eff}} = 1/n$. Then the expression for the entropy turns out to be in agreement with the formula (11.10), corresponding to the first FLRW equation (11.11) when the Casimir energy reaches the bound $E_{\text{C}} = E_{\text{BH}}$.

Inhomogeneous EoS fluid and bulk viscosity

Let us now explore the case of an $n{+}1$ -dimensional Universe filled with a fluid satisfying an inhomogeneous EoS. This kind of EoS, generalizing the perfect fluid model, has been considered in previous chapters as a way to describe effectively the dark energy. We assume an EoS expressed as a function of the scale factor,

$$p = w(a)\rho + g(a) \ . \tag{11.20}$$

This EoS fluid could be taken to correspond to modified gravity, or to bulk viscosity. By introducing (11.20) in the energy conservation equation (11.3) we obtain

$$\rho'(a) + \frac{n(1+w(a))}{a}\rho(a) = -n\frac{g(a)}{a} \ . \tag{11.21}$$

Here we have performed a variable change $t = t(a)$ such that the prime over ρ denotes derivative with respect to the scale factor a . The general solution of this equation is

$$\rho(a) = \mathrm{e}^{-F(a)}\left(K - n\int \mathrm{e}^{F(a)}\frac{g(a)}{a}da\right) \quad \text{where} \quad F(a) = \int^a \frac{1+w(a')}{a'}da' \ , \tag{11.22}$$

and K is an integration constant. As shown above, only for some special choices of the functions $w(a)$ and $g(a)$, the formula (11.10) can be recovered. Let us assume, as an example, that $w(a) = -1$ and $g(a) = -a^m$, with $m =$ constant. Then, the energy density behaves as $\rho \propto a^m$. Hence, by following the same steps as described above, the extensive and subextensive energy go as a^{m+n} , and by imposing conformal invariance and the rescaling properties (11.6), we calculate the dependence on the entropy to be

$$E_{\text{E}} = \frac{\alpha}{4\pi n a^{-(m+n)}} S^{-m/n} \ , \quad E_{\text{C}} = \frac{\beta}{4\pi n a^{-(m+n)}} S^{-(2n+m/n)} \ . \tag{11.23}$$

The expression for the entropy is easily constructed by combining these two expressions and substituting the extensive part by the total energy. This gives us the same expression as in (11.9) with $w = -(n{+}m)/n$. Note that for $m = -(1+n)$, the formula (11.10) is recovered and also its correspondence with the CFT formula. However for a generic power m , the CV formula cannot be reconstructed, like the cases studied above. Only for some special choices does the correspondence work, leading to the identification between the FLRW equation and the Cardy formula.

11.2 On the cosmological bounds near future singularities

In Ref.[297], Verlinde proposed a new universal bound on cosmology based on a restriction of the Casimir energy E_C ; cf. his entropy formula (11.10). This new bound postulated was

$$E_{\rm C} \leq E_{\rm BH} \ , \tag{11.24}$$

where $E_{\rm BH} = n(n-1)\frac{V}{8\pi G a^2}$. It was deduced by the fact that in the limit when the Universe passes between strongly and weakly self-gravitating regimes, the Bekenstein entropy $S_{\rm B} = \frac{2\pi a}{n} E$ and the Bekenstein-Hawking entropy $S_{\rm BH} = (n-1)\frac{V}{4Ga}$, which define each regime, are equal. This bound could be interpreted to mean that the Casimir energy never becomes able to reach sufficient energy, $E_{\rm BH}$, to form a black hole of the size of the Universe. It is easy to verify that the strong ($Ha \geq 1$) and weak ($Ha \leq 1$) self-gravity regimes have the following restrictions on the total energy,

$$\begin{aligned} E &\leq E_{\rm BH} \quad \text{for} \quad Ha \leq 1 \ , \\ E &\geq E_{\rm BH} \quad \text{for} \quad Ha \geq 1 \ . \end{aligned} \tag{11.25}$$

From here it is easy to calculate the bounds on the entropy of the Universe in the case when the Verlinde formula (11.10) is valid; this is (as shown in the above sections) for an effective radiation dominated Universe $w_{\rm eff} \sim 1/n$. The bounds for the entropy deduced in Ref.[297] for $k = 1$ are

$$\begin{aligned} S &\leq S_{\rm B} \quad \text{for} \quad Ha \leq 1 \ , \\ S &\geq S_{\rm H} \quad \text{for} \quad Ha \geq 1 \ , \end{aligned} \tag{11.26}$$

where $S_{\rm B}$ is the Bekenstein entropy defined above, and $S_{\rm H}$ is the Hubble entropy given by (11.11). Note that for the strong self-gravity regime, $Ha \geq 1$, the energy range is $E_{\rm C} \leq E_{\rm BH} \leq E$. According to the formula (11.10) the maximum entropy is reached when the bound is saturated, $E_{\rm C} = E_{\rm BH}$. Then $S = S_{\rm H}$, such that the FLRW equation coincides with the CV formula, thus indicating a connection with CFT. For the weak regime, $Ha \leq 1$, the range of energies goes as $E_{\rm C} \leq E \leq E_{\rm BH}$ and the maximum entropy is reached earlier, when $E_{\rm C} = E$, yielding the result $S = S_{\rm B}$. The entropy bounds can be extended to more general cases, corresponding to an arbitrary EoS parameter w . By taking the bound (11.24) to be universally valid one can easily deduce the new entropy bounds for each regime, from the expression of the entropy (11.9). These new bounds, discussed in Ref.[308], differ from the ones given in (11.26), but still establish a bound on the entropy as long as the bound on $E_{\rm C}$ expressed in (11.24) is taken to be valid. The entropy bounds can be related through the first FLRW equation, yielding the following quadratic expression (for $k = 1$),

$$S_{\rm H}^2 + (S_{\rm B} - S_{\rm BH})^2 = S_{\rm B}^2 \ . \tag{11.27}$$

We would like to study what happens to the bounds, particularly to the fundamental bound (11.24), when the cosmic evolution is close to a future singularity; then the effective fluid dominating the cosmic evolution could have an unusual EoS. As shown below, for some class of future singularities such a bound could soften the singularities in order to avoid violation of the *universal* bound (11.24). It could be interpreted to mean that quantum effects become important when the bound is reached. However, as the violation of the bound could happen long before the singularity even in the presence of quantum effects, it could be a signal of breaking of the universality of the bound (11.24). Let us first of all give a list of the possible future cosmic singularities, which can be classified according to Ref.[246] as

- Type I ("Big Rip"): For $t \to t_s$, $a \to \infty$ and $\rho \to \infty$, $|p| \to \infty$.
- Type II ("Sudden"): For $t \to t_s$, $a \to a_s$ and $\rho \to \rho_s$, $|p| \to \infty$.

- Type III: For $t \to t_s$, $a \to a_s$ and $\rho \to \infty$, $|p| \to \infty$.
- Type IV: For $t \to t_s$, $a \to a_s$ and $\rho \to \rho_s$, $p \to p_s$ but higher derivatives of Hubble parameter diverge.

Note that the above list was suggested in the case of a flat FLRW Universe. As we consider in this chapter a closed Universe ($k = 1$), we should make an analysis to see if the list of singularities given above is also valid in this case. It is straightforward to see that all the singularities listed above can be reproduced for a particular choice of the effective EoS. To show how the cosmic bounds behave for each type of singularity, we could write an explicit solution of the FLRW equations, expressed as a function of time depending on free parameters that will be fixed for each kind of singularity. Then, the Hubble parameter may be written as follows

$$H(t) = \sqrt{\frac{16\pi G}{n(n-1)}\rho - \frac{1}{a^2}} = H_1(t_s - t)^m + H_0 \ , \tag{11.28}$$

where m is a constant properly chosen for each type of singularity. Note that this is just a solution that ends in the singularities mentioned above, but there are other solutions which also reproduce such singularities. We will study how the cosmic bounds behave near each singularity listed above. As pointed out in Ref.[133, 246], around a singularity quantum effects could become important as the curvature of the Universe grows and diverges in some of the cases. In other words, approaching the finite-time future singularity the curvature grows and universe reminds the early universe where quantum gravity effects are dominant ones because of extreme conditions. Then one has to take into account the role of such quantum gravity effects which should define the behaviour of the universe just before the singularity. Moreover, they may act so that to prevent the singularity occurrence. In a sense, one sees the return of quantum gravity era. However, the consistent quantum gravity theory does not exists so far. Then, in order to estimate the influence of quantum effects to universe near to singularity one can use the effective action formulation. We will apply the effective action produced by conformal anomaly (equivalently, the effective fluid with pressure/energy-density corresponding to conformal anomaly ones) because of several reasons. It is known that at high energy region (large curvature) the conformal invariance is restored so one can neglect the masses. Moreover, one can use large N approximation to justify why large number of quantum fields may be considered as effective quantum gravity. Finally, in the account of quantum effects via conformal anomaly we keep explicitly the graviton (spin 2) contribution. The conformal anomaly T_A has the following well-known form

$$T_A = b\left(F + \frac{2}{3}\Box R\right) + b'G + b''\Box R \ . \tag{11.29}$$

Here we assume for simplicity a 3+1 dimensional spacetime. Then, F is the square of a 4D Weyl tensor and G is the Gauss-Bonnet invariant,

$$F = \frac{1}{3}R^2 - 2R_{ij}R^{ij} + R_{ijkl}R^{ijkl} \ , \quad G = R^2 - 4R_{ij}R^{ij} + R_{ijkl}R^{ijkl} \, . \tag{11.30}$$

The coefficients b and b' in (11.29) are described by the number of N scalars, $N_{1/2}$ spinors, N_1 vector fields, N_2 gravitons and $N_{\rm HD}$ higher derivative conformal scalars. They can be written as

$$b = \frac{N + 6N_{1/2} + 12N_1 + 611N_2 - 8N_{\rm HD}}{120(4\pi)^2} \ , \quad b' = -\frac{N + 11N_{1/2} + 62N_1 + 1411N_2 - 28N_{\rm HD}}{360(4\pi)^2} \ . \tag{11.31}$$

As b'' is arbitrary it can be shifted by a finite renormalization of the local counterterm. The conformal anomaly T_A can be written as $T_A = -\rho_A + 3p_A$, where ρ_A and p_A are the energy and pressure densities

respectively. By using (11.29) and the energy conservation equation $\rho_A + 3H(\rho_A + p_A) = 0$, one obtains the following expression for ρ_A [217, 246],

$$\begin{aligned}\rho_A &= -\frac{1}{a^4}\int dta^4 HT_A \\ &= -\frac{1}{a^4}\int dta^4 H\left[-12b\dot{H}^2 + 24b'(-\dot{H}^2 + H^2\dot{H} + H^4) - (4b+6b'')(\dddot{H} + 7H\ddot{H} + 4\dot{H}^2 + 12H^2\dot{H})\right]\end{aligned} \tag{11.32}$$

The quantum corrected FLRW equation is given by

$$H^2 = \frac{8\pi G}{3}(\rho + \rho_A) - \frac{1}{a^2}\ . \tag{11.33}$$

We study now how the bounds behave around the singularity in the classical case when no quantum effects are added, and then include the conformal anomaly (11.29) quantum effects in the FLRW equations. We will see that for some cases the violation of the cosmic bound can be avoided.

Big Rip Singularity

This type of singularity has been very well studied and has become very popular as it is a direct consequence in the majority of the cases when the effective EoS parameter is less than -1, the so-called phantom case. Observations currently indicate that the phantom barrier could have already been crossed or it will be crossed in the near future, so a lot of attention has been paid to this case. It can be characterized by the solution (11.28) with $m \leq -1$, and this yields the following dependence of the total energy density on the scale factor near the singularity, when $a \gg 1$, for a closed Universe ($k = 1$),

$$\rho = \frac{n(n-1)}{16\pi G}H^2 + \frac{1}{a^2} \sim a^{-n(1+w)} \quad \text{for} \quad t \to t_s\ , \tag{11.34}$$

where we have chosen $H_1 = 2/n|1+w|$ with $w < -1$, $m = -1$ and $H_0 = 0$ for clarity. This solution drives the Universe to a Big Rip singularity for $t \to t_s$, where the scale factor diverges. If the singularity takes place, the bound (11.24) has to be violated before this happens. This can be seen from equation (11.34), as the Casimir energy behaves as $E_{\rm C} \propto a^{n|w|}$ while the Bekenstein-Hawking energy goes as $E_{\rm BH} \propto a^{n-2}$. Then, as $w < -1$, the Casimir energy grows faster than the BH energy, so close to the singularity where the scale factor becomes very big, the value of $E_{\rm C}$ will be much higher than $E_{\rm B}$, thus violating the bound (11.24). Following the postulate from Ref.[297] one could interpret the bound (11.24) as the limit where General Relativity and Quantum Field Theory converge, such that when the bound is saturated quantum gravity effects should become important. QG corrections could help to avoid the violation of the bound and may be the Big Rip singularity occurrence. As this is just a postulate based on the CV formula, which is only valid for special cases as shown in the sections above, the bound on $E_{\rm C}$ could not be valid for any kind of fluid.

Let us now include the conformal anomaly (11.29) as a quantum effect that becomes important around the Big Rip. In such a case there is a phase transition and the Hubble evolution will be given by the solution of the FLRW equation (11.33). Let us approximate to get some qualitative results, assuming $3+1$ dimensions. Around t_s the curvature is large, and $|\rho_A| >> (3/\kappa^2)H^2 + k/a^2$. Then $\rho \sim -\rho_A$, and from (11.32)we get

$$\dot{\rho} + 4H\rho = H\left[-12b\dot{H}^2 + 24b'(-\dot{H}^2 + H^2\dot{H} + H^4) - (4b+6b'')(\dddot{H} + 7H\ddot{H} + 4\dot{H}^2 + 12H^2\dot{H})\right]\ . \tag{11.35}$$

We assume that the energy density, which diverges in the classical case, behaves now as

$$\rho \sim (t_s - t)^\lambda\ , \tag{11.36}$$

where λ is some negative number. By using the energy conservation equation $\dot{\rho} + 3H(1+w)\rho = 0$, the Hubble parameter goes as $H \sim 1/(t_s - t)$. We can check if this assumption is correct in the presence of quantum effects by inserting both results in Eq. (11.35). We get

$$\rho \sim 3H^4(-13b + 24b') \ . \tag{11.37}$$

Hence as $b > 0$ and $b' < 0$, ρ becomes negative, which is an unphysical result. Thus ρ should not go to infinity in the presence of the quantum correction. This is the same result as obtained in Ref.[246] where numerical analysis showed that the singularity is moderated by the conformal anomaly, so that the violation of the bound that naturally occurs in the classical case can be avoided/postponed when quantum effects are included.

Sudden singularity

This kind of singularity is also problematic with respect to the bounds, but as the energy density ρ does not diverge, the violation of the bound may be avoided for some special choices. The sudden singularity can be described by the solution (11.28) with $0 < m < 1$, and constants $H_{0,1} > 0$. Then the scale factor goes as

$$a(t) \propto \exp\left[-\frac{H_1}{m+1}(t_s - t)^{m+1} + H_0 t\right] \ , \tag{11.38}$$

which gives $a(t) \sim \mathrm{e}^{h_0 t}$ (de Sitter) close to t_s . From the first FLRW equation the total energy density becomes

$$\rho = H^2(t) + \frac{1}{a^2} = [H_1(t_s - t)^m + H_0]^2 + \exp\left[2\frac{H_1}{m+1}(t_s - t)^{m+1} - 2H_0 t\right] \ , \tag{11.39}$$

which tends to a constant $\rho \sim H_0^2 + \mathrm{e}^{-2H_0 t_s}$ for $t \to t_s$. Then the Casimir energy grows as $E_{\mathrm{C}} \propto H_0^2 a^n + a^{n-2}$, while $E_{\mathrm{BH}} \propto a^{n-2}$ close to t_s . The BH energy grows slower than the Casimir energy, and the bound is violated for a finite t . However, by an specific choice of the coefficients, the violation of the bound (11.24) could be avoided. For $H_0 = 0$, and by some specific coefficients, the bound could be obeyed. In general, it is very possible that E_{C} exceeds its bound. In the presence of quantum corrections, the singularity can be avoided but the bound can still be violated, depending on the free parameters for each model. We may assume that in the presence of the conformal anomaly for $n = 3$, the energy density grows as [246]

$$\rho = \rho_0 + \rho_1(t_s - t)^{\lambda} \ , \tag{11.40}$$

where ρ_0 and ρ_1 are constants, and λ is now a positive number. Then the divergences on the higher derivatives of the Hubble parameter can be avoided, as is shown in Ref.[246]. Nevertheless, E_{C} still grows faster than E_{BH} , such that the Universe has to be smaller than a critical size in order to hold the bound (11.24) as is pointed in Ref.[308] for the case of a vacuum dominated universe.

Type III singularity

This type of singularity is very similar to the Big Rip, in spite of the scale factor $a(t)$ being finite at the singularity. The solution (11.28) reproduces this singularity by taking $-1 < m < 0$. The scale factor goes as

$$a(t) = a_s \exp\left[-\frac{H_1}{m+1}(t_s - t)^{m+1}\right] \ , \tag{11.41}$$

where for simplicity we take $H_0 = 0$. Then, for $t \to t_s$, the scale factor $a(t) \to a_s$. To see how $E_{\rm C}$ behaves near the singularity, let us write it in terms of the time instead of the scale factor,

$$E_{\rm C} \propto a_s^n H_1^2 (t_s - t)^{2m} + a_s^{n-2} \ , \tag{11.42}$$

where $m < 0$. Hence, the Casimir energy diverges at the singularity, while $E_{\rm BH} \propto a_s^{n-2}$ takes a finite value for the singularity time t_s, so the bound is clearly violated long before the singularity. Then, in order to maintain the validity of the bound (11.24), one might assume, as in the Big Rip case, that GR is not valid near or at the bound. Even if quantum effects are included, as was pointed in Ref.[246], for this type of singularity the energy density diverges more rapidly than in the classical case, so that the bound is also violated in the presence of quantum effects.

Type IV singularity

For this singularity, the Hubble rate behaves as

$$H = H_1(t) + (t_s - t)^\alpha H_2(t) \ . \tag{11.43}$$

Here $H_1(t)$ and $H_2(t)$ are regular function and do not vanish at $t = t_s$. The constant α is not integer and larger than 1. Then the scale factor behaves as

$$\ln a(t) \sim \int dt H_1(t) + \int dt \, (t_s - t)^\alpha H_2(t) \ . \tag{11.44}$$

Near $t = t_s$, the first term dominates and every quantities like ρ, p, and a etc. are finite and therefore the bound (11.24) would not be violated near the singularity.

Big Bang singularity

When the matter with $w \geq 0$ coupled with gravity and dominates, the scale factor behaves as

$$a \sim t^{\frac{2}{n(1+w)}} \ . \tag{11.45}$$

Then there appears a singularity at $t = 0$, which may be a Big Bang singularity. Although the Big Bang singularity is not a future singularity, we may consider the bound (11.24) when $t \sim 0$. Since $n(1+w) > 2$, the energy density behaves as $\rho \sim a^{-n(1+w)}$ and therefore the Casimir energy behaves as $E_{\rm C} \sim a^{-nw}$. On the other hand, we find $E_{\rm BH} \sim a^{n-2}$. Then when $n > 2$ or when $n \geq 2$ and $w > 0$, $E_{\rm C}$ dominates when $a \to 0$, that is, when $t \to 0$, and the bound (11.24) is violated. This tells us, as expected, that quantum effects become important in the early universe.

Above, we have thus explored what happens near the future cosmic singularities. We have seen that in general, and with some very special exceptions on the case of Type II and Type IV, the bound will be violated if one assumes the validity of GR close to the singularity. Even if quantum corrections are assumed, it seems that the bound will be violated, although in the Big Rip case the singularity may be avoided when quantum effects are incorporated. It is natural to suggest, in accordance with Verlinde, that the bound on the Casimir energy means a finite range for the validity of the classical theory. When this kind of theory becomes saturated, some other new quantum gravity effects have to be taken into account. We conclude that the universality of the bound (11.24) is not clear and may hold just for some specific cases, like the radiation dominated Universe.

11.3 $F(R)$-gravity and the Cardy-Verlinde formula

We specify here a modified $F(R)$-gravity modeled as an effective fluid and construct the corresponding CV formula for it. The action that describes $F(R)$-gravity is given by

$$S = \frac{1}{2\kappa^2}\int d^{n+1}x\sqrt{-g}(F(R) + L_m) \ , \tag{11.46}$$

where L_m represents the matter Lagrangian and $\kappa^2 = 8\pi G$. The field equations are obtained by varying the action (11.46) with respect to the metric $g_{\mu\nu}$,

$$R_{\mu\nu}F'(R) - \frac{1}{2}g_{\mu\nu}F(R) + g_{\mu\nu}\Box F'(R) - \nabla_\mu\nabla_\nu F'(R) = \kappa^2 T^{(m)}_{\mu\nu} \ . \tag{11.47}$$

Here $T^{(m)}_{\mu\nu}$ is the energy-momentum tensor for the matter filling the Universe, and we have assumed a $1+3$ spacetime for simplicity. For closed $3+1$ FLRW Universe, the modified FLRW equations are expressed as

$$\begin{aligned} \frac{1}{2}F(R) - 3(H^2+\dot{H})F'(R) + 3HF''(R)\dot{R} &= \kappa^2\rho_m \ , \\ -\frac{1}{2}F(R) + \left[3H^2 + \dot{H} + \frac{2}{a^2}\right]F'(R) - [(\partial_{tt}F'(R)) + 2H(\partial_t F'(R))] &= \kappa^2 p_m \ , \end{aligned} \tag{11.48}$$

where primes denote derivatives respect to R and dots with respect to t. These equations can be rewritten in order to be comparable with those of standard GR. For such a propose the geometric terms can be presented as an effective energy-density $\rho_{F(R)}$ and a pressure $p_{F(R)}$,

$$\begin{aligned} H^2 + \frac{1}{a^2} &= \frac{\kappa^2}{3F'(R)}\rho_m + \frac{1}{3F'(R)}\left[\frac{RF'(R)-F(R)}{2} - 3H\dot{R}F''(R)\right] \ , \\ 2\dot{H} + 3H^2 + \frac{1}{a^2} &= -\frac{\kappa^2}{F'(R)}p_m - \frac{1}{F'(R)}\left[\dot{R}^2F'''(R) + 2H\dot{R}F''(R) + \ddot{R}F''(R) + \frac{1}{2}(F(R) - RF'(R))\right] \end{aligned} \tag{11.49}$$

Then, an EoS for the geometric terms can be defined as $p_{F(R)} = w_{F(R)}\rho_{F(R)}$. We can define an effective energy-density $\rho = \rho_m/F'(R) + \rho_{F(R)}$ and pressure $p = p_m/F'(R) + p_{F(R)}$. Hence, for some special cases the formula for the entropy developed in the second section can be obtained in $F(R)$-gravity (for an early attempt deriving a CV formula in a specific version of $F(R)$-gravity, see [59]). For example, for an $F(R)$ whose solution gives $\rho \propto a^{-3(1+w_{\rm eff})}$, the formula for the entropy (11.9) is recovered although in general, as in the cases studied above, no such expression can be given. On the other hand, one could assume that the geometric terms do not contribute to the matter sector. Supposing a constant EoS matter fluid, the expression for the entropy is given by (11.9), although the cosmic Cardy formula (11.11) has not the same form and in virtue of the modified first FLRW equation (11.48) the form of the Hubble entropy $S_{\rm H}$, the total energy E, and the Bekenstein energy $E_{\rm BH}$, will be very different. It is not easy to establish correspondence between two such approaches. Note that using the effective fluid representation the generalized CV formula may be constructed for any modified gravity.

Now we consider the case where $F(R)$ behaves as

$$F(R) \sim R^\alpha \ , \tag{11.50}$$

when the curvature is small or large. Then if the matter has the EoS parameter $w > -1$, by solving (11.48) we find

$$a \sim \left\{ \begin{array}{cc} t^{\frac{2\alpha}{n(1+w)}} & \text{when } \frac{2\alpha}{n(1+w)} > 0 \\ (t_s - t)^{\frac{2\alpha}{n(1+w)}} & \text{when } \frac{2\alpha}{n(1+w)} < 0 \end{array} \right. \ . \tag{11.51}$$

Then there may appear a singularity at $t = 0$, which corresponds to the Big Bang singularity, or at $t = t_s$, which corresponds to the Big Rip singularity. Since the Casimir energy behaves as $E_{\rm C} \sim a^{-nw}$ but $E_{\rm BH} \sim a^{n-2}$, only when $t \to 0$, $E_{\rm C}$ dominates in case that $n > 2$ and $w \geq 0$ or in case that $n \geq 2$ and $w > 0$. Even in the phantom phase where $\frac{2\alpha}{n(1+w)} < 0$, the bound (11.24) is not violated.
Let us now consider de Sitter space solution in $F(R)$-gravity (for review of CV formula in dS or AdS spaces, see [65, 66]). As was pointed in the above part of the present thesis, almost every function $F(R)$ admits a de Sitter solution. In this case the formula for the entropy (11.9) can be reproduced for $w = -1$, and even the universal bound (11.24) can hold by taking a critical size of the Universe. The formula that relates the cosmic bounds in (11.27) is easily obtained also in $F(R)$-gravity for a de Sitter solution. In such a case one can identify

$$S_{\rm H} = \frac{H_0 V}{2G} \ , \quad S_{\rm B} = \frac{aV}{24G}\frac{F(R_0)}{F'(R_0)} \ , \quad S_{\rm BH} = \frac{V}{2Ga} \ , \tag{11.52}$$

which corresponds to the first FLRW equation written as $S_{\rm H}^2 + (S_{\rm B} - S_{\rm BH})^2 = S_{\rm B}^2$. Thus, one can conclude that dynamical entropy bounds are not violated for modified gravity with de Sitter solutions. Note that quantum gravity effects may be presented also as an effective fluid contribution. In case when de Sitter space turns out to be the solution, even with the account of quantum gravity the above results indicate that dynamical cosmological/entropy bounds are valid. In other words, the argument indicates the universality of dynamical bounds. It seems that their violation is caused only by future singularities if they are not cured by quantum gravity effects. Note that a large number of modified gravity theories do not contain future singularities; they are cured by higher derivatives terms.

11.4 Discussions

In summary, we have derived a generalized CV formula for multicomponent, interacting fluids, generalized in the sense that an inhomogeneous EoS was assumed. We also considered modified $F(R)$-gravity, using its fluid representation. We showed that for some special cases the formula is reduced to the standard CV formula expressing the correspondence with 2d CFT theory. The dynamical entropy bound for all above cases was found. The universality of dynamical entropy bound near all four types of the future singularity, as well as the initial Big Bang singularity, was investigated. It was proved that except from some special cases of Type II and Type IV singularity the dynamical entropy bound is violated near the singularity. Taking into account quantum effects of conformally invariant matter does not improve the situation.

One might think that the dynamical entropy bound is universal and that its violation simply indicates that the situation will be changed with the introduction of quantum gravity effects. However, arguments given below indicate that it is not the case and that the future singularity is the domain where all known physical laws and equations are not valid. Indeed, taking account of quantum effects such as done in section VI does not improve the situation with respect to non-universality of the dynamical entropy bound. From another side, it was shown that the dynamical entropy bound is valid for the de Sitter solution. Having in mind that quantum gravity corrections may always be presented as a generalized effective fluid, one sees that the dynamical cosmological bound is not valid near the singularity (even when account is taken of Quantum Gravity). It is only when modified gravity (with or without quantum corrections) is regular in the future, where the future universe is asymptotically de Sitter, that the dynamical bound remains valid. Hence, the problem of non-universality of dynamical entropy bound is related to the more fundamental question about the real occurrence of a future singularity. It remains a challenge to find any observational indications for the structure of the future universe.

Chapter 12

Casimir effects near Big Rip singularity

[1]Here, it is investigated the analytical properties of the cosmic expansion, especially near Big Rip singularity, for a comic fluid with and without bulk viscosity, and with an oscillating equation of state. We study the effects of the inclusion of a Casimir-induced term in the energy-momentum tensor, and its behavior near the future singularity. Cosmic fluids - whether considered in the early or in the late epochs - are usually taken to be nonviscous, but there are two viscosity (shear and bulk) coefficients naturally occurring in general linear hydrodynamics; the linear approximation meaning physically that one is considering only first order deviations from thermal equilibrium. The shear viscosity coefficient is evidently of importance when dealing with flow near solid surfaces, but it can be crucial also under boundary-free conditions such as in isotropic turbulence. In later years it has become more common to take into account viscosity properties of the cosmic fluid, however. Because of assumed spatial isotropy in the fluid the shear viscosity is usually left out; any anisotropic deviations like those encountered in the Kasner universe become rather quickly smoothed out. Thus only the bulk viscosity coefficient, called ζ , remains in the energy-momentum tensor of the fluid. One should here note, however, that at least in the plasma region in the early universe the value of the shear viscosity as derived from kinetic theory is greater than the bulk viscosity by many orders of magnitude. Cf., for instance, Refs. [54, 64].

Early treatises on viscous cosmology are given in Refs. [162, 250]. Also in [97], it is considered viscous dark energy and phantom evolution using Eckhart's theory of irreversible thermodynamics. As discussed in Refs. [75, 225], a dark fluid with a time dependent bulk viscosity can be considered as a fluid with an inhomogeneous equation of state. Some other recent papers on viscous cosmology are Refs. [150, 197].

A special branch of viscous cosmology is to investigate how the bulk viscosity can influence the future singularity, commonly called the Big Rip, when the fluid is in the phantom state corresponding to the thermodynamic parameter w being less than -1 . Some recent papers in this direction are Refs. [43, 45, 46, 50, 52, 163]. In particular, as first pointed out in Ref. [48], the presence of a bulk viscosity proportional to the expansion can cause the fluid to pass from the quintessence region into the phantom region and thereby inevitably lead to a future singularity.

The purpose of the present chapter is to generalize these viscous cosmology theories in the sense that we take into account the *Casimir effect*. We shall model the Casimir influence in a different way than in

[1]This Chapter is based on: [51, 161]

the preceding chapter, by writing the total Casimir energy in the same form as for a perfectly conducting shell, identifying the cosmological "shell" radius essentially with the cosmic scale factor. This is a very simple, though natural, approach to the problem. The approach is of the same kind as that followed in an earlier quantum cosmology paper dealing with the expanding FLRW universe in the nonviscous case [56]. Other ways of treating the influence from the Casimir effect in cosmology; the reader may consult, for instance, Refs. [124, 130, 134, 156, 277].

12.1 Formalism

We start with the standard spatially flat FLRW metric. We let subscript 'in' refer to present time quantities, and choose $t_{in} = 0$. The scale factor $a(t)$ is normalized such that $a(0) \equiv a_{in} = 1$. The equation of state is taken as

$$p = w\rho, \tag{12.1}$$

with w constant. As mentioned, $w < -1$ in the phantom region (ordinary matter is not included in the model). The bulk viscosity ζ is taken to be constant. The energy-momentum tensor of the fluid can be expressed as

$$T_{\mu\nu} = \rho U_\mu U_\nu + \tilde{p}(g_{\mu\nu} + U_\mu U_\nu)\ , \tag{12.2}$$

where U_μ is the comoving four-velocity and $\tilde{p} = p - \zeta\theta$ is the effective pressure, $\theta = 3H = 3\dot{a}/a$ being the scalar expansion. The Friedmann equations take the form

$$\theta^2 = 24\pi G\rho, \tag{12.3}$$

$$\dot{\theta} + \frac{1}{6}\theta^2 = -4\pi G\tilde{p}. \tag{12.4}$$

The energy conservation equation leads to

$$\dot{\rho} + (\rho + p)\theta = \zeta\theta^2. \tag{12.5}$$

From the above equations the differential equation for the scalar expansion, with w and ζ as free parameters, can be derived as

$$\dot{\theta} + \frac{1}{2}(1+w)\theta^2 - 12\pi G\zeta\theta = 0. \tag{12.6}$$

The solution is, when subscript zero signifies that the Casimir effect is so far left out,

$$\theta_0(t) = \frac{\theta_{in} e^{t/t_c}}{1 + \frac{1}{2}(1+w)\theta_{in} t_c (e^{t/t_c} - 1)}\ . \tag{12.7}$$

Here θ_{in} is the initial (present-time) scalar expansion and t_c the 'viscosity time'

$$t_c = \frac{1}{12\pi G\zeta}. \tag{12.8}$$

Since $(1+w) < 0$ by assumption, it follows from Eq. (12.7) that the future singularity occurs at a rip time t_{s0} given by

$$t_{s0} = t_c \ln\left[1 - \frac{2}{1+w}\frac{1}{\theta_{in} t_c}\right]. \tag{12.9}$$

This means that in the nonviscous limit $\zeta \to 0$,

$$t_{s0} = -\frac{2}{1+w}\frac{1}{\theta_{in}}, \tag{12.10}$$

which is independent of ζ . At the other extreme, in the high viscosity limit $t_c \to 0$,

$$t_{s0} = t_c \ln\left[\frac{-2}{1+w}\frac{1}{\theta_{in}t_c}\right], \tag{12.11}$$

showing that t_{s0} becomes small. The fluid is quickly driven into the Big Rip singularity.

12.2 The Casimir effect included

As mentioned above, a simple and natural way of dealing with the Casimir effect in cosmology is to relate it to the single length parameter in the ($k=0$) theory, namely the scale factor a . It means effectively that we should put the Casimir energy E_c inversely proportional to a . This is in accordance with the basic property of the Casimir energy, viz. that it is a measure of the stress in the region interior to the "shell" as compared with the unstressed region on the outside. The effect is evidently largest in the beginning of the universe's evolution, when a is small. At late times, when $a \to \infty$, the Casimir influence should be expected to fade away. As we have chosen a nondimensional, we shall introduce an auxiliary length L in the formalism. Thus we adopt in model in which

$$E_c = \frac{C}{2La}, \tag{12.12}$$

where C is a nondimensional constant. This is the same form as encountered for the case of a perfectly conducting shell [205]. In the last-mentioned case, C was found to have the value

$$C = 0.09235. \tag{12.13}$$

The expression (12.12) is of the same form as adopted in Ref. [55, 57]). It is strongly related to the assumptions made by Verlinde when dealing with the holographic principle in the universe [297], and explored in the previous chapter.

In the following we shall for definiteness assume C to be positive, corresponding to a repulsive Casimir force, though C will not necessarily be required to have the value (12.13). The repulsiveness is a characteristic feature of conducting shell Casimir theory, following from electromagnetic field theory under the assumption that dispersive short-range effects are left out ([55, 205]). Another assumption that we shall make, is that C is small compared with unity. This is physically reasonable, in view of the conventional feebleness of the Casimir force.

The expression (12.12) corresponds to a Casimir pressure

$$p_c = \frac{-1}{4\pi(La)^2}\frac{\partial E_c}{\partial(La)} = \frac{C}{4\pi L^4 a^4}, \tag{12.14}$$

and leads consequently to a Casimir energy density $\rho_c \propto 1/a^4$.

The Casimir energy-momentum tensor

$$T^c_{\mu\nu} = \rho_c U_\mu U_\nu + p_c(g_{\mu\nu} + U_\mu U_\nu), \tag{12.15}$$

together with the Casimir equation of state $p_c = w_c\rho_c$, yield the energy balance

$$\frac{\dot{\rho}_c}{\rho_c} + (1+w_c)\theta = 0, \tag{12.16}$$

having the solution $\rho_c a^{3(1+w_c)}$ = constant. To get $\rho_c \propto 1/a^4$ we must have $w_c = 1/3$. The Casimir contributions to the pressure and energy density become accordingly

$$p_c = \frac{C}{8\pi L^4 a^4}, \quad \rho_c = \frac{3C}{8\pi L^4 a^4}. \tag{12.17}$$

The Friedmann equations now become

$$\theta^2 = 24\pi G \left(\rho + \frac{3C}{8\pi L^4 a^4} \right), \tag{12.18}$$

$$\dot{\theta} + \frac{1}{6}\theta^2 = -4\pi G \left(w\rho - \zeta\theta + \frac{C}{8\pi L^4 a^4} \right), \tag{12.19}$$

while the energy conservation equation preserves its form,

$$\dot{\rho} + (1+w)\rho\theta = \zeta\theta^2. \tag{12.20}$$

Note again that we are considering the *dark energy fluid* only, with density ρ and thermodynamical parameter w. The ordinary matter fluid (dust) is left out.

Solving ρ from Eq. (12.18) and inserting into Eq. (12.20) we obtain as governing equation for the scalar expansion

$$\dot{\theta} + \frac{1}{2}(1+w)\theta^2 - 12\pi G\zeta\theta = -(1-3w)\frac{3GC}{2L^4 a^4}\ . \tag{12.21}$$

It is convenient to introduce the constant α, defined as

$$\alpha = -(1+w) > 0, \tag{12.22}$$

and also to define the quantity $X(t)$,

$$X(t) = 1 - \frac{1}{2}\alpha\theta_{in}t_c(e^{t/t_c} - 1), \tag{12.23}$$

which satisfies

$$X(0) = 1, \quad X(t_{s0}) = 0. \tag{12.24}$$

In view of the assumed smallness of C we now make a Stokes expansion for θ to the first order,

$$\theta(t) = \theta_0(t) + C\theta_1(t) + O(C^2), \tag{12.25}$$

using henceforth the convention that subscript zero refers to the $C = 0$ case. The zeroth order solution is

$$\theta_0(t) = \theta_{in}X^{-1}e^{t/t_c}, \tag{12.26}$$

in accordance with Eq. (12.7). It corresponds to the zeroth order scale factor

$$a_0(t) = X^{-\frac{2}{3\alpha}}, \tag{12.27}$$

satisfying $a_0(0) = 1$ as before.

As the right hand side of Eq. (12.21) is already of order C, we can replace $a(t)$ with $a_0(t)$ in the denominator. Thus we get the following equation for the first order correction coefficient θ_1:

$$\dot{\theta}_1 - \left(\alpha\theta_{in}X^{-1}e^{t/t_c} + 12\pi G\zeta \right)\theta_1 = -(1-3w)\frac{3G}{2L^4}X^{\frac{8}{3\alpha}}. \tag{12.28}$$

We impose the same initial condition for the scalar expansion as in the $C=0$ case: $\theta(0)=\theta_0(0)\equiv\theta_{in}$. It means according to Eq. (12.25) that $\theta_1(0)=0$.

The homogeneous solution of Eq. (12.28), called θ_{1h}, may be written

$$\theta_{1h}(t)=\exp\left[\int_0^t(\alpha\theta_{in}X^{-1}e^{t/t_c}+12\pi G\zeta)dt\right], \tag{12.29}$$

satisfying $\theta_{1h}(0)=1$. The the full solution becomes

$$\theta_1(t)=-(1-3w)\frac{3G}{2L^4}\theta_{1h}(t)\cdot\int_0^t\frac{X^{\frac{8}{3\alpha}}}{\theta_{1h}}dt, \tag{12.30}$$

satisfying $\theta_1(0)=0$. The two terms on the right hand side of Eq. (12.25), $\theta_0(t)$ and $C\theta_1(t)$, are accordingly determined.

12.2.1 Analytic approximation for low viscosity fluid

Although in general the expression for $\theta(t)$ has to be calculated numerically, the main features of the solution can be shown already analytically. Consistent with the assumed smallness of C we need not distinguish between the rip time t_s corresponding to $C\neq0$ and the rip time t_{s0} corresponding to $C=0$. Let us assume for mathematical simplicity the low-viscosity limit

$$\theta_{in}t_c\gg1, \tag{12.31}$$

being physically the most important case also. It corresponds to $t_{s0}/t_c=2/(\alpha\theta_{in}t_c)\ll1$. Then,

$$X(t)\sim\frac{1}{2}\alpha\theta_{in}(t_{s0}-t)=\frac{t_{s0}-t}{t_{s0}}, \tag{12.32}$$

$$\theta_0(t)\sim\frac{2}{\alpha}\frac{1}{t_{s0}-t}, \tag{12.33}$$

$$a_0(t)\sim\left(\frac{t_{s0}}{t_{s0}-t}\right)^{\frac{2}{3\alpha}}. \tag{12.34}$$

Both $\theta_0(t)$ and $a_0(t)$ diverge ($\alpha>0$ by assumption). Using Eq. (12.32) we can calculate $\theta_1(t)$ from Eq. (12.30),

$$\theta_1(t)=-(1-3w)\frac{9G}{2L^4}\frac{\alpha t_{s0}}{8+9\alpha}\frac{1-(1-t/t_{s0})^{\frac{8}{3\alpha}+3}}{(1-t/t_{s0})^2}. \tag{12.35}$$

From the expansion (12.25) we thus obtain for the scalar expansion to the first order in C,

$$\theta(t)=\frac{2}{\alpha t_{s0}}\frac{1}{1-t/t_{s0}}\left\{1-(1-3w)\frac{9GC}{4L^4}\frac{\alpha^2t_{s0}^2}{8+9\alpha}\frac{1-(1-t/t_{s0})^{\frac{8}{3\alpha}+3}}{1-t/t_{s0}}\right\}. \tag{12.36}$$

The viscosity is absent in this expression. This is as we would expect in view of the low-viscosity approximation.

The expression (12.36) cannot, however, be valid near the singularity. The reason is that the Taylor expansion in C in Eq. (12.25) is not applicable at $t=t_{s0}$. The solution (12.36) can be applied safely

as long as t stays considerably smaller than t_{s0}. By making a first order expansion in t/t_{s0} of the expression between the curly parentheses we can write the solution in simplified form as

$$\theta(t) = \frac{2}{\alpha t_{s0}} \frac{1}{1 - t/t_{s0}} \left\{ 1 - (1 - 3w) \frac{3GC\alpha t_{s0}}{4L^4} t \right\}, \quad t \ll t_{s0}. \tag{12.37}$$

As $(1 - 3w) > 0$ this means that $\theta(t)$ becomes slightly reduced by the Casimir term. The repulsive Casimir force causes the energy density ρ in Eq. (12.18) to be slightly smaller than in the $C = 0$ case.

To deal with the conditions close to the singularity, we have to go back to the governing equations themselves.

Let now $t_{\zeta s}$ denote the singularity time in the presence of viscosity. We thus get

$$t_{s\zeta} = t_c \ln \left(1 + \frac{2}{\alpha} \frac{1}{\theta_{in} t_c} \right). \tag{12.38}$$

It corresponds to $\theta(t_{s\zeta}) = \infty$. We see that $t_{s\zeta}$ is always less than the singularity time for the nonviscous case,

$$t_{s\zeta} < t_{s,\zeta=0} = \frac{2}{\alpha\, \theta_{in}}. \tag{12.39}$$

For the scalar expansion we find close to the singularity, again assuming for simplicity low viscosity so that $\theta_{in} t_c \gg 1$ [49],

$$\theta(t) = \frac{2/\alpha}{t_{s\zeta} - t}, \quad t \to t_{s\zeta}. \tag{12.40}$$

In turn, this corresponds to

$$a(t) \sim \frac{1}{(t_{s\zeta} - t)^{2/3\alpha}}, \quad t \to t_{s\zeta}, \tag{12.41}$$

$$\rho(t) \sim \frac{1}{(t_{s\zeta} - t)^2}, \quad t \to t_{s\zeta}. \tag{12.42}$$

12.2.2 On the nonviscous case

It may finally be worthwhile to consider the entirely nonviscous case, while keeping $C > 0$. Let us analyze first of all, a simple example when $w(t) = w_0$ is a constant. Setting $\zeta = 0$ we get from Eq. (12.21) the governing equation for the scale factor a:

$$a^3 \ddot{a} + \frac{1}{2}(1 + 3w) a^2 \dot{a}^2 = -(1 - 3w) \frac{GC}{2L^4}. \tag{12.43}$$

And the general solution for this equation results:

$$t - t_s = \int \frac{a da}{\sqrt{k a^{1-3w_0} + \frac{GC}{L^4}}}, \tag{12.44}$$

where k is an integration constant. When we have a phantom field with $w_0 << -1$, the solution takes the form:

$$a(t) = B \left[\left(1 + \frac{2}{e^{\sqrt{kB}(t_s - t)} - 1} \right)^2 - 1 \right]^{1/1-3w_0}, \tag{12.45}$$

here $B = \frac{GC}{kL^4}$. Note that for $t \to t_s$, the scale factor given by (12.45) diverges, such that the so-called Big Rip singularity takes place. On the other hand, if we neglect the Casimir contribution $0 < C << 1$, the solution for the scale factor yields

$$a(t) \sim (t_s - t)^{\frac{2}{3}(1+w_0)} = (t_s - t)^{-\frac{2}{3}|1+w_0|} \ , \tag{12.46}$$

which is the solution in absence of the Casimir term. This solution grows slower than in the above case when the Casimir contribution is taken into account (12.45). Also note that the Rip time t_s for each case is different if the Casimir contribution is neglected in the second case, due to t_s depends on the content from the Universe as it can be seen by the first Friedmann equation $t_s - t_0 = \int_{a_0}^{\infty} \frac{da}{H_0 a(\sum_i \Omega^i(a))}$, where t_0 denotes the current time and $\Omega^i(a)$ is the cosmological parameter for the component i of the Universe Let us now to analyze the equation (12.21) with null viscosity when $w = w(t)$ is a periodic function given by,

$$w = -1 + w_0 \cos \omega t \ . \tag{12.47}$$

By inserting (12.47) into (12.21), the equation turns much more complicated than the constant case studied above. As the Casimir contribution is assumed to be very small, we can deal the equation (12.21) by perturbation methods. Hence, the solution might be written as

$$\theta(t) = \theta_0 + C\theta_1 + O(C^2) \ . \tag{12.48}$$

And by inserting (12.48) into (12.21), the zero and first order in perturbations can be separated, and it yields the following two equations:

$$\dot{\theta_0} + \frac{1}{2} w_0 \cos \omega t \theta_0 = 0 \ ,$$

$$\dot{\theta_1} + w_0 \cos \omega t \theta_0 \theta_1 = -\frac{3G}{2L^4}(4w_0 \cos \omega t)\frac{1}{a_0^4(t)} \ , \tag{12.49}$$

where $a_0(t) = \exp\left(\int dt \frac{2\omega}{3(w_1 + w_0 \sin \omega t)}\right)$ is the solution for the zero order. We suppose for simplicity $w_1 \sim w_0$, such that a Big Rip singularity appears when $1 + \sin \omega t_s = 0$. As we are interested in the possible effects close to the singularity, we can expand the trigonometric functions as series around t_s . Then, the second equation in (12.49) takes the form:

$$\dot{\theta_1} + \frac{4}{3\omega^2}\frac{1}{t_s - t}\theta_1 =$$

$$= -\frac{3G}{2L^4}(4 + \frac{3w_0}{\omega}(t_s - t))\mathrm{e}^{-\frac{16}{3\omega(t_s - t)}} \ , \tag{12.50}$$

whose solution is given by

$$\theta_1(t) = k(t_s - t)^{\frac{4}{3\omega^2}} + (t_s - t)^{\frac{4}{3\omega^2}} g(1/t_s - t) \ ,$$

where

$$g(x) = \frac{3GC}{2L^4 B}\left[4x^A \mathrm{e}^{-Bx} - \frac{4(4A+B)}{B}\int x^{A-1}\mathrm{e}^{-Bx}dx\right] \ . \tag{12.51}$$

Here $A = \frac{4-6\omega^2}{3\omega^2}$ and $B = \frac{16}{3\omega}$. Then, we can analyze the behavior from the contribution to the solution at first order (12.51) when the Universe is close to the singularity. When $t \to t_s$ ($x \to \infty$), the function (12.51) goes to zero, what means that the Casimir contribution has no effect at the Rip time.

Thus, we have demonstrated, that dynamical Casimir effect gives no essential contribution to (phantom oscillating) dark energy dynamics near to Rip singularity.

12.3 Concluding remarks

Considerable attention has recently been devoted to the behavior of the dark energy fluid near the future singularity. Various possibilities have been contemplated. In addition to the references given above, we may refer also to the papers [72, 220]. It has even been suggested that the future singularity can be avoided via quantum gravity effects. Thus in Ref. [133] it is shown how the universe may turn into a de Sitter phase.

Let us finally summarize our results above:

- If $\zeta > 0$ and $C = 0$, the rip singularity time t_{s0} is given by Eq. (12.9). In particular, if $\zeta \to 0$ then Eq. (12.10) holds.

- If $\zeta > 0$ and $C > 0$, the scalar expansion $\theta(t)$ is given by the first-order series (12.25), with $\theta_0(t)$ and $\theta_1(t)$ given by Eqs. (12.26) and (12.30). In the low-viscosity case $\theta_{in} t_c \gg 1$, $\theta(t)$ is given by the series (12.36) which, however, is not applicable near the singularity as $\theta(t)$ is not analytic in C at the singularity.

- Near the singularity, the Casimir effect fades away and the viscous rip time $t_{s\zeta}$ is given by Eq. (12.38). Corresponding values for $\theta(t), a(t)$ and $\rho(t)$ near the singularity follow from Eqs. (12.40) - (12.42).

- If $\zeta = 0$ and $C > 0$, the Rip time is usually longer than in the viscous case. The singularity occurs even with the presence of the Casimir term, which does not affect to the evolution, as it can be seen in the examples studied.

Chapter 13

Conclusions and perspectives

Let us summarize and analyze the models and results discussed along the present thesis. Possible perspectives for the future theoretical physics in cosmology, in the frame of the theories studied here, are discussed here. As commented in the introduction, the main aim of this work is to show how different approaches could resolve the problem of dark energy and shape the entire evolution of the history of the Universe, from inflation to current epoch. The different answers and intrinsic questions related to each model have been analyzed, as none of them is free of its proper unsolved questions. The possibility to distinguish between different theories is discussed, where the observations as well as possible predictions have to play a fundamental role.

First of all, we have discussed cosmological models where scalar fields (non-)minimally coupling are included. It has been shown that a single scalar-field, with a canonical or phantom kinetic term, is enough to shape the entire Universe evolution, where the appearance of different accelerated epochs, inflation and current phase, looks to be a natural consequence of the behavior of the scalar field. In this sense, several examples have been provided, where different solutions for the Hubble parameter is explored, from de Sitter-like solutions to solutions with a rapidly dynamical EoS for the scalar field. In general, models with phantom fields contain future singularities, specially the so-called Big Rip singularity, where the scale factor diverges. It has been shown that a phantom scalar field could own, among other microscopical problems that are not discussed here, divergent instabilities during the transition from canonical to phantom phase due to the null kinetic term at the phantom barrier, where a discontinuity is produced. This problem can be cured by introducing more than one scalar field, where each field behaves as canonical or phantom but they do not have a transition phase. Then, with several scalar fields the unification of late-time acceleration and inflation is achieved, and even when the effective EoS parameter for the Universe crosses the phantom barrier. The reconstruction of the inflationary epoch when there is more than one scalar field has been performed in detail, where a type of slow-roll inflation is reconstructed, what provides constraints on the free parameters of the model.
In theories of the type of Brans-Dicke theory, where an scalar field is coupling directly to the Ricci scalar in the action, we have shown that dark energy can also be easily reproduced in this way. This kind of theories can lead to severe corrections of the Newtonian law at local scales. Nevertheless, this kind of problems can be avoided by the chameleon mechanism. In this sense, we have considered here a coupling of the type $1 + f(\phi)$, where the function $f(\phi)$ can be constrained to be null at local scales (and GR is recovered) by means of the chameleon mechanism. Another important purpose for studying these theories was to compare the solutions in the original (Jordan) frame and in the Einstein frame, both related by a conformal transformation. Several examples are provided, and it is shown a very interesting example,

where pure de Sitter cosmology in the Einstein frame, transforms into a completely different solution (as expected) in the Jordan frame, where the solution contains a singularity. Hence, we can see that a regular solution in one frame does not correspond to a regular one in the other, but singular points of the spacetime in the other frame can be obtained.
On the other hand, oscillating cosmologies have been explored by effective dark fluids (those with non-static and inhomogeneous EoS that can be effective descriptions of scalar fields or modified gravities). It is shown that the current accelerated era can be seen as a phase of the Universe evolution that is periodically repeated, and which started with the birth of the Universe and the period of inflation. This could provide an answer to the so-called "coincidence problem" as it predicts a periodicity of the behavior of the expansion, whose period can be fitted with the estimated age of the Universe. The end could be an eternal oscillating Universe, or depending on the parameters of the model, a final future singularity. It has been shown that such behavior can be well implemented by scalar fields or modified gravities. Also interacting dark energy is considered, at least in an effective way with no microscopic details, where a proper interaction could drive the accelerated expansion, such that a signal of a possible interaction between dust matter and an unknown dark energy fluid could be distinguished from other models.
Of course, many questions could be discussed in greater detail, a more realistic matter content should be taken into account, and the universe expansion history described in a more precise and detailed way. After all, we live in an era of more and more precise cosmological tests. Nevertheless, the effective description of the cosmic expansion history presented here in terms of scalar fields or effective dark fluids seems quite promising. Using it in more realistic contexts in which, of course, technical details become much more complicated, appears to be quite possible. Nevertheless, scalar fields seem more as a toy or effective descriptions models than realistic models.

Non-minimally scalar-tensor theories with a null kinetic term are mathematically equivalent to $f(R)$ gravities (though this is surely to be restricted to the classical level). This modification of GR, whose starting point is a generalization of the Hilbert-Einstein action, can easily reproduced any cosmic solution. Here, we have shown that from a scalar-tensor theory, there is a corresponded $f(R)$ action, such that the reconstruction of modified gravity theories of this type can be easily achieved given a cosmic solution and using an auxiliary scalar field. The main success of $f(R)$ gravities is that it can naturally explain the dark energy epoch with no need of an extra exotic field, and even could act relaxing the vacuum energy expected value, what could give an answer to the cosmological constant problem. Several examples of this kind of reconstruction have been given, where unification of inflation and late-time acceleration seems to be quite possible in $f(R)$ gravities. One has to point out that in higher order gravities, the presence of several de Sitter solutions is quite natural, such that accelerated phases along the Universe evolution can be seen as normal epochs that depending on each de Sitter point could be a (un)stable phase.
By using of a new reconstruction technique, based on rewriting the FLRW equations as functions of the number of e-foldings instead of the cosmic time, several realistic cosmologies have been reconstructed. It has been shown that an exact solution of Hubble parameter, corresponded to an evolution of the kind of Λ CDM model, yields an action composed by a divergent hypergeometric function of the Ricci scalar and a cosmological constant, what actually is reduced for the physical case to the Einstein-Hilbert action plus a cc, what means that exact Λ CDM model can be reproduced physically only by GR plus a cosmological constant, as it is natural. However, actions can be constructed for $f(R)$ gravity, which in spite of not providing an exact Λ CDM behavior for the Hubble parameter, can mimic a cosmological constant at the current epoch (low redshift). One of the examples provided here, shows that after a matter dominated epoch, the Universe enters in an accelerated expansion phase where the effective EoS parameter $w \sim -1$, and then it crosses the phantom barrier ending in a future Big Rip singularity.
It is well known that higher order theories of gravity may lead to important corrections to gravitational law at local scales (Ref. [287]). It is remarkable that even particular modified gravity does not fulfill some cosmological bounds. This may be always achieved using new reconstruction techniques via a corresponding reconstruction at very early universe. Nevertheless, it has been shown that the so-called viable models

can avoid problems at small scales, where GR is recovered, such that the effects of higher order terms in the field equations become important at cosmological scales (some examples of viable $f(R)$ gravities can be found in Refs.[88, 176, 234, 286]). In this sense, implementations of this type of models can be reconstructed as it has been shown using the same technique, where some corrections to these viable models can be introduced to incorporate observational requirements, and not only unified cosmological history is reproduced but may also achieve to explain dark matter effects (see Ref. [84]).
One type of these viable models has been studied, where it is showed that can reproduce quite well inflation and late-time acceleration with a power-law phase between them (radiation/matter dominated epochs). Even, it was shown that de Sitter solution for inflation can be unstable providing a successful exit from this early phase. At low redshifts the cosmological parameter associated to the extra terms in the action behaves very similarly to the one by a cosmological constant, but where the effective EoS parameter associated to these geometrical terms has a dynamical behavior, as it is natural, moving from zero at redshifts $z \sim 1.4$ to -1 for redshifts $z \sim 0$, and it mimics a cosmological constant at the current epoch. In the same model, the effects of the presence of a phantom fluid in the model have been explored.
On the other hand, it is well known that phantom scalar models in the Einstein frame do not have a physical correspondence in the Jordan frame, where $F(R)$ gravity is defined. Nevertheless, the presence of a phantom fluid can solve this problem, where the scalar field in the Einstein frame owns a canonical kinetic term, and the phantom behavior corresponds solely to the fluid.
Also modified gravities where Gauss-Bonnet invariants are evolved have been studied. As equivalent to $f(R)$ theories, Gauss-Bonnet gravity can easily reproduce unification of inflation and late-time acceleration. A model has been considered, where the action is given by $R+f(G)$, i.e. GR action plus a correction coming from a function of the Gauss-Bonnet invariant. This kind of models can easily reproduce Λ CDM model with no need of dark energy. Also the pure Gauss-Bonnet gravity has been explored, where several examples of important cosmological solutions have been given. Finally, general models of the type $f(R, G)$ have been considered, where similar results are obtained.
Hence, we have showed that modified gravities can be well reconstructed to reproduce the entire cosmological evolution and keep unchanged the behavior of gravity at local scales, where GR is well tested. Probably the next step in this study would be to know how they differ from other kinds of models. The study of the evolution of density perturbations may provide some differences that could be measured in the future (see Ref. [120]). Even in the regions of the spacetime where the curvature is very large, as in the neighborhood of a black hole, some corrections on the solution from GR could be found (see Refs. [87, 121]).

The recently proposal Hořava-Lifshitz gravity has been also studied, which seems to be power-counting renormalizable by losing the invariance under full diffeomorphisms. In an equivalent way as the extension from GR to $f(R)$ gravity, HL gravity has been extended to more general actions. A generalization of HL action shows that unification of accelerated epochs of Universe evolution can be performed. In this sense, reconstruction of cosmological solutions have been explored showing that any cosmic solution can be reconstructed from a given modification of HL gravity $f(\tilde{R})$. Even it seems that the so-called viable models in standard $f(R)$ gravity can be extended to HL gravity, where its properties remained unchanged: inflation and dark energy epochs are well reproduced while the scalar mode related to $f(\tilde{R})$ is neglible in the Newtonian corrections. The construction of an action free from future singularities has been performed. Also the study on the stability of solutions of the FLRW equations have been performed, where important information about the restrictions of the gravitational theories is obtained. It was found that $f(\tilde{R})$ gravity can well explain the end of the matter dominated epoch and reproduce the entire cosmic history. Then, the theory can well reproduce the Universe evolution and it seems a good promise as a candidate for a quantum field theory of gravity, in spite of the fact that serious problems are unresolved yet. The breakdown of Lorentz invariance supposes that different observers will measure different speeds of light as well as the field equations will be different for all of them. However, it is assumed that in the IR limit the full diffeomorphism is recovered $z = 1$. The main problem of the theory is the existence of a spin-0 mode that introduces severe instabilities around the Mikowski background. However, it has been shown here

that for some specific actions $f(\tilde{R})$, one can find unstable flat solutions but stable de Sitter spacetime, which can be considered as the natural vacuum solution of the theory instead of Mikowski spacetime. On the other hand, a different approach has been performed in order to recover general covariance in the IR limit (see Ref. [175]) by means of the incorporation of a $U(1)$ symmetry into the theory. Here we have extended such theory to more general actions, where the incorporation of higher terms can account for the effects of cosmic expansion.
Hence, we have shown that in the frame of HL gravity, accelerated expansion can be well reproduced as well as other features coming from standard $f(R)$ gravity. However, as it was pointed out, HL gravities contain several problems, and in spite of the new proposals, the problem to recover general covariance in a natural way does not have a consistent solution yet, but interesting attempts have been explored here. Also, as any new science, predictive power has to be provided, so that the breakdown of covariance should be measured in some limit. Nevertheless, it seems that the energy scales dealt with in the laboratories are far from being able to probe this.

Finally, in the last part of the thesis, future singularities have been studied, where semiclassical effects are taken into account to show its possible effects near the singularities. Accounting for a cosmological Casimir effect seems not to cure the occurrence of the Big Rip singularity, but it could affect the cosmological evolution as an effective radiation fluid, although its density probably has decayed nowadays. Even by taking into account the conformal anomaly, it does not seem to affect future singularities. Moreover, it has been proven that for a cosmology that contains future singularities, universal bounds on the entropy are violated, even much before reaching the singularity, what predicts the non-validity of such bounds.
On the other hand, generalizations of the CV formula have been performed, where it seems that its correspondence with a 2d CFT is only valid for some special cases, and can not be generalized. Nevertheless, the CV formula gives a direct way to the derivation of dynamical bounds for universe and black hole entropy bounds.

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